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SETS OF "NON-TYPICAL" POINTS HAVE FULL TOPOLOGICAL ENTROPY AND FULL HAUSDORFF DIMENSION

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ABSTRACT

For subshifts of finite type, conformal repellers, and conformal horseshoes, we prove that the set of points where the pointwise dimensions, local entropies, Lyapunov exponents, and Birkhoff averages do not exist simultaneously, carries **full** topological entropy and **full** Hausdorff dimension. This follows from a much stronger statement formulated for a class of symbolic dynamical systems which includes subshifts with the specification property. Our proofs strongly rely on the multifractal analysis of dynamical systems and constitute a non-trivial mathematical application of this theory.

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1. Introduction

In the numerical study of a dynamical system one is often interested in the asymptotic behavior of "typical" points, with respect to some invariant measure. This study gives important information about the observable properties of the dynamical system, and "typical" points with respect to different measures give complementary information.

The set of "non-typical" points, i.e., the set of points that is "typical" with respect to no measure, has rarely been considered in the literature. In this paper, we show that, surprisingly, in several situations central in the theory of dynamical systems this set contains complete information about some observable properties. Namely, the set of "non-typical" points carries **full** topological entropy and **full** Hausdorff dimension.

In order to prove this statement, we combine "typical" points with respect to different invariant measures to produce sets of "non-typical" points which still carry information about the measures. The proof strongly relies on the multifractal analysis of dynamical systems. An important element of unification in our approach is the use of Carathéodory dimension characteristics, introduced by Pesin.

2. Subshifts of finite type

2.1. PRELIMINARIES. Let σ: $\{1,\ldots,p\}^{\mathbb{N}} \to \{1,\ldots,p\}^{\mathbb{N}}$ be the shift map given by $\sigma(i_0 i_1 \cdots) = (i_1 i_2 \cdots)$. We fix a number $\beta > 1$ and define a metric on $\{1,\ldots,p\}^{\mathbb{N}}$ by

$$d((i_0 i_1 \cdots), (j_0 j_1 \cdots)) = \sum_{k=0}^{\infty} \beta^{-k} |i_k - j_k|. \tag{1}$$

The space $\{1,\ldots,p\}^{\mathbb{N}}$ is compact with respect to this metric.

For every compact subset $\Sigma \subset \{1,\ldots,p\}^{\mathbb{N}}$ such that $\sigma\Sigma \subset \Sigma$ we consider the **subshift** $\sigma|\Sigma$, and denote its topological entropy by $h(\sigma) = h(\sigma|\Sigma)$. Let A be a $p \times p$ matrix all of whose entries a_{ij} are either 0 or 1. We consider the compact subset $\Sigma = \Sigma_A \subset \{1,\ldots,p\}^{\mathbb{N}}$ composed of the sequences $(i_0 i_1 \cdots)$ such that $a_{i_n i_{n+1}} = 1$ for every $n \geq 0$. The map $\sigma|\Sigma_A$ is called the **subshift of finite type** with **transfer matrix** A. We recall that $\sigma|\Sigma_A$ is topologically mixing if and only if there is a positive integer k such that all entries of A^k are positive. We recall that $h(\sigma|\Sigma_A) = \log \rho(A)$, where $\rho(A)$ denotes the spectral radius of A.

2.2. IRREGULAR SETS FOR BIRKHOFF AVERAGES. Let $C(\Sigma)$ be the space of continuous functions on Σ. For each function $g \in C(\Sigma)$, we define the **irregular**

set for the Birkhoff averages of g by

$$\mathfrak{B}(g) = \left\{x \in \Sigma\colon \lim_{n\to\infty} \tfrac{1}{n} S_n g(x) \text{ does not exist}\right\},$$

where

$$S_n g(x) = \sum_{k=0}^{n} g(\sigma^k x) \tag{2}$$

for each $x \in \Sigma$ and $n \in \mathbb{N}$. By the Birkhoff Ergodic Theorem, $\mu(\mathfrak{B}(g)) = 0$ for every σ-invariant measure μ.

We say that g_1 and g_2 are **cohomologous** if $g_1 - g_2 = \psi - \psi \circ \sigma + c$, for some $\psi \in C(\Sigma)$ and $c \in \mathbb{R}$. If g_1 and g_2 are cohomologous, then $\mathfrak{B}(g_1) = \mathfrak{B}(g_2)$ and $c = P(g_1) - P(g_2)$, where P denotes the topological pressure with respect to σ (see, for example, [13] for the definition).

We say that a set $\Lambda \subset \Sigma$ is **σ-invariant** if $\Sigma \cap \sigma^{-1}\Lambda = \Lambda$. The set $\mathfrak{B}(g)$ is σ-invariant but is in general not compact. The notion of topological entropy for non-compact sets was introduced by Bowen in [7]. Later it was considered by Pesin and Pitskel' in [14] with an approach closer to the one we use. The following statement shows that the zero measure set $\mathfrak{B}(g)$ is "observable"; namely, $\mathfrak{B}(g)$ carries full topological entropy for a large class of functions g.

THEOREM 2.1: *Let $\sigma|\Sigma$ be a topologically mixing subshift of finite type, and $g_1, \ldots, g_k$ Hölder continuous functions on Σ. Then the following properties are equivalent:*

1. *the functions $g_1, \ldots, g_k$ are non-cohomologous to 0;*
2. *$\mathfrak{B}(g_1) \cap \cdots \cap \mathfrak{B}(g_k)$ is non-empty;*
3. *$\mathfrak{B}(g_1) \cap \cdots \cap \mathfrak{B}(g_k)$ is a proper dense subset;*
4. *$h(\sigma|\mathfrak{B}(g_1) \cap \cdots \cap \mathfrak{B}(g_k)) > 0$;*
5. *$h(\sigma|\mathfrak{B}(g_1) \cap \cdots \cap \mathfrak{B}(g_k)) = h(\sigma)$.*

If g is cohomologous to 0, then $\mathfrak{B}(g) = \emptyset$. Thus, Property 1 in Theorem 2.1 follows immediately from each of the Properties 2, 3, 4, and 5. Since any non-empty invariant set of a topologically mixing **one-sided** shift is dense, Property 3 is an immediate consequence of Property 2. In particular, given an arbitrary point $x \in \Sigma$, the set $\bigcup_{n=1}^{\infty} \sigma^{-n}x$ composed by all the preimages of x is σ-invariant and thus is dense in Σ. Theorem 2.1 follows from a much more general statement in Theorem 7.1 below. An announcement of Theorem 2.1 appeared in [6] for the case $k = 1$.

Let $C_\theta(\Sigma)$ be the space of Hölder continuous functions on Σ with Hölder exponent θ. For a function $\phi \in C_\theta(\Sigma)$ we define its norm by

$$\|\phi\|_\theta = \sup\{|\phi(x)|\colon x \in \Sigma\} + \inf\{K > 0\colon \phi \in C_\theta^K(\Sigma)\}, \tag{3}$$

where $C_\theta^K(\Sigma)$ is the family of functions

$$\{\phi \in C_\theta(\Sigma)\colon |\phi(x)-\phi(y)| \le Kd(x,y)^\theta \text{ for every } x,\, y \in \Sigma\}.$$

The following statement shows that plenty Hölder continuous functions are non-cohomologous to 0.

PROPOSITION 2.2: *If $\sigma|\Sigma$ is a topologically mixing subshift of finite type, then the following properties hold:*

1. *The family of Hölder continuous functions on Σ which are non-cohomologous to 0 contains a dense subset of $C(\Sigma)$ (with respect to the supremum norm).*
2. *For each $\theta \in (0,1)$ the family of functions in $C_\theta(\Sigma)$ which are non-cohomologous to 0 contains an open dense subset of $C_\theta(\Sigma)$.*

By Theorem 2.1 and Proposition 2.2, if $\sigma|\Sigma$ is a topologically mixing subshift of finite type, then $h(\sigma|\mathfrak{B}(g)) = h(\sigma)$ for g out of a dense family in $C(\Sigma)$.

For each integer $k \ge 0$ and tuple $(i_0 \cdots i_k) \in \{1,\dots,p\}^{k+1}$, we define a **cylinder** of **length** $k+1$ by $\{(j_0 j_1 \cdots) \in \Sigma\colon (j_0 \cdots j_k) = (i_0 \cdots i_k)\}$. Let L be the family of non-constant linear combinations of characteristic functions of cylinders (of arbitrary length). It is clear that L is a dense family composed by Hölder continuous functions. Proposition 2.2 is a consequence of the following.

PROPOSITION 2.3: *If $\sigma|\Sigma$ is a topologically mixing subshift of finite type, then the following properties hold:*

1. *The family L contains a subset of functions non-cohomologous to 0 which is dense in $C(\Sigma)$ (with respect to the supremum norm).*
2. *For each $\theta \in (0,1)$, the family $L \cap C_\theta(\Sigma)$ contains a subset of functions non-cohomologous to 0 which is dense in $C_\theta(\Sigma)$.*

We also consider the set

$$\mathfrak{B} = \left\{x \in \Sigma\colon \lim_{n\to\infty} \frac{1}{n} S_n g(x) \text{ does not exist for some } g \in C(\Sigma)\right\}. \tag{4}$$

Note that

$$\mathfrak{B} = \bigcup_{g \in C(\Sigma)} \mathfrak{B}(g).$$

For a topologically mixing subshift of finite type, it follows from Theorem 2.1 that

$$h(\sigma|\mathfrak{B}) = h(\sigma).$$

This formula was first established by Pesin and Pitskel' in [14] in the case of the Bernoulli shift on two symbols. Their methods of proof are different from ours; moreover, it is not clear if their proof can be generalized to arbitrary subshifts of finite type.

2.3. Generic points. Let $\mathfrak{M}$ be the family of σ-invariant Borel probability measures on Σ. Given $\mu \in \mathfrak{M}$, the point $x \in \Sigma$ is called a **generic point for** μ if

$$\lim_{n\to\infty} \frac{1}{n} S_n g(x) = \int_\Sigma g\, d\mu$$

for every $g \in C(\Sigma)$. We denote the set of generic points for μ by $\mathfrak{G}(\mu)$. Clearly, $\mathfrak{B} \subset \Sigma \setminus \bigcup_{\mu\in\mathfrak{M}} \mathfrak{G}(\mu)$. If $x \in \Sigma \setminus \mathfrak{B}$, then the map

$$g \mapsto \lim_{n\to\infty} \frac{1}{n} S_n g(x)$$

defines a σ-invariant bounded linear functional on $C(\Sigma)$, and, by the Riesz Representation Theorem, $x \in \mathfrak{G}(\mu)$ for some σ-invariant measure μ. Hence,

$$\mathfrak{B} = \Sigma \setminus \bigcup_{\mu\in\mathfrak{M}} \mathfrak{G}(\mu).$$

Remarks: 1. If μ is σ-invariant, then $\mu(\mathfrak{B}) = 0$ (by the Birkhoff Ergodic Theorem and the separability of $C(\Sigma)$).

2. If $\sigma|\Sigma$ is uniquely ergodic, then the set $\mathfrak{B}$ is empty.

2.4. Irregular sets for local entropies. For each probability measure μ on Σ, we define the **irregular set for the local entropies of** μ by

$$\mathfrak{H}(\mu) = \left\{ x \in \Sigma\colon \lim_{n\to\infty} -\frac{\log \mu(C_n(x))}{n} \text{ does not exist} \right\},$$

where $C_n(x)$ denotes the cylinder of length n which contains the point $x \in \Sigma$. The set $\mathfrak{H}(\mu)$ is σ-invariant but may not be compact.

Remarks: 1. If μ is σ-invariant, then $\mu(\mathfrak{H}(\mu)) = 0$ (using the Shannon–McMillan–Breiman theorem).

2. If μ is a Gibbs measure, then there is a cohomology class of functions in $C(\Sigma)$ such that $\mathfrak{H}(\mu) = \mathfrak{B}(g)$ if and only if g belongs to this cohomology class.
3. Let μ be a σ-invariant measure of maximal entropy. If μ is a Gibbs measure, then $\mathfrak{H}(\mu)$ is empty. For example, if $\sigma|\Sigma$ is a topologically mixing subshift with the specification property (in particular, if $\sigma|\Sigma$ is a subshift of finite

type or a sofic subshift), then any σ-invariant measure of maximal entropy is a Gibbs measure.

We denote by $G(\sigma|\Sigma)$ the family of Gibbs measures on Σ (with respect to σ) having a Hölder continuous potential. For a topologically mixing subshift of finite type, it follows from Theorem 2.1 that if $\mu \in G(\sigma|\Sigma)$ is not the measure of maximal entropy, then

$$h(\sigma|\mathfrak{H}(\mu)) = h(\sigma). \tag{5}$$

We describe the relationship between $\mathfrak{B}$ and the irregular sets $\mathfrak{H}(\mu)$.

PROPOSITION 2.4: *For a topologically mixing subshift of finite type, we have*

$$\mathfrak{B} = \bigcup_{\mu:\ \mu \text{ is a Gibbs measure}} \mathfrak{H}(\mu). \tag{6}$$

Remarks: 1. All the statements in this section remain true when we substitute the one-sided shift $\sigma\colon \{1,\dots,p\}^{\mathbb{N}} \to \{1,\dots,p\}^{\mathbb{N}}$ by the two-sided shift $\sigma\colon \{1,\dots,p\}^{\mathbb{Z}} \to \{1,\dots,p\}^{\mathbb{Z}}$.

2. With the special type of metric defined in (1), we have $h(\sigma|Z) = \dim_H Z \cdot \log\beta$ for any subset $Z \subset \Sigma$, where $\dim_H Z$ denotes the Hausdorff dimension of Z. Thus, by (5), for a topologically mixing subshift of finite type $\sigma|\Sigma$, and a measure $\mu \in G(\sigma|\Sigma)$ which is not the measure of maximal entropy, we have

$$\dim_H \mathfrak{H}(\mu) = \dim_H \Sigma.$$

3. Repellers

Let $f\colon M \to M$ be a C^1 map of a smooth manifold, and J an f-invariant compact subset of M. We say that f is **expanding** on J and that J is a **repeller** of f if there are constants $C > 0$ and $\beta > 1$ such that $\|d_x f^n u\| \geq C\beta^n \|u\|$ for all $x \in J$, $u \in T_x M$, and $n \geq 1$.

It is well known that repellers admit Markov partitions of arbitrarily small diameter. Each Markov partition has associated a one-sided subshift of finite type $\sigma|\Sigma$, and a **coding map** $\chi\colon \Sigma \to J$ for the repeller, which is Hölder continuous, onto, and satisfies $f \circ \chi = \chi \circ \sigma$ and $\sup\{\text{card}(\chi^{-1}x)\colon x \in J\} < \infty$ (see, for example, [13] for details).

A differentiable map $f\colon M \to M$ is called **conformal** on a set J if $d_x f$ is a multiple of an isometry at every point $x \in J$. Well-known examples of conformal expanding maps include one-dimensional Markov maps and holomorphic maps. We write $a(x) = \|d_x f\|$ for each $x \in M$. For a repeller J of a conformal $C^{1+\varepsilon}$ expanding map f, the equilibrium measure m_D of $-\dim_H J \cdot \log a$ on J is called the **measure of maximal dimension** (m_D is the unique f-invariant measure μ such that $\dim_H \mu = \dim_H J$; see, for example, [13] for details). We denote by m_E the **measure of maximal entropy**, i.e., the equilibrium measure of 0.

Recall that $\dim_H Z$ denotes the Hausdorff dimension of the set Z. We denote by $P_Z(g)$ the topological pressure of the continuous function g on the set Z (which is not necessarily compact or f-invariant); see Section 6.1 and [13] for details. The following statement is a consequence of the proof of Theorem 2.1 in [1].

THEOREM 3.1: *Let J be a repeller of a topologically mixing C^1 expanding map f such that f is conformal on J. For every subset $Z \subset J$ (not necessarily compact or f-invariant), we have $\dim_H Z = s$, where s is the unique root of the equation $P_Z(-s\log a) = 0$.*

Set $\mathfrak{B}_f = \chi(\mathfrak{B})$ and $\mathfrak{H}_f(\mu) = \chi(\mathfrak{H}(\mu))$. We observe that

$$\mathfrak{B}_f \supset \left\{x \in J\colon \lim_{n\to\infty} \frac{1}{n} S_n g(x) \text{ does not exist for some } g \in C(J)\right\},$$

where $S_n g$ is defined by (2). Let J be a subset of M. We define the **irregular set for the Lyapunov exponents of f** by

$$\mathfrak{L}_f = \left\{x \in J\colon \lim_{n\to\infty} \frac{1}{n} \log \|d_x f^n\| \text{ does not exist}\right\},$$

and for each probability measure μ on J, the **irregular set for the pointwise dimensions of μ** by

$$\mathfrak{D}(\mu) = \left\{x \in J\colon \lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \text{ does not exist}\right\},$$

where $B(x,r) \subset J$ is the ball of radius r centered at x. It is easy to see that

$$\lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} = \lim_{n\to\infty} -\frac{\log \mu(B(x,e^{-n}))}{n}, \tag{7}$$

whenever any of the limits exist. In a similar way to that in Section 2.2, for a repeller J of a topologically mixing expanding map, if any of the invariant sets $\mathfrak{B}_f$, $\mathfrak{H}_f(\mu)$, $\mathfrak{L}_f$, and $\mathfrak{D}(\mu)$ is non-empty, then it is dense in J.

By Kingman's Subadditive Ergodic Theorem, we have $\mu(\mathfrak{L}_f) = 0$ for any f-invariant probability measure μ on J. In [18], Schmeling and Troubetzkoy proved that if μ is a measure invariant under an expanding map (not necessarily conformal), then $\mu(\mathfrak{D}(\mu)) = 0$ (they also consider the case of expanding maps with singularities, and invariant measures concentrated "outside the singularities"; see [18]).

We now describe several irregular sets which carry full topological entropy and full Hausdorff dimension. Recall that $G(f|J)$ denotes the family of Gibbs measures on J (with respect to f) having a Hölder continuous potential.

THEOREM 3.2: *If J is a repeller of a topologically mixing C^1 expanding map f, then the following properties hold:*

1. $h(f|\mathfrak{B}_f) = h(f|J)$;
2. *if* $\mu \in G(f|J)$ *and* $\mu \neq m_E$, *then* $\mathfrak{H}_f(\mu) \subset \mathfrak{B}_f$ *and* $h(f|\mathfrak{H}_f(\mu)) = h(f|J)$.

The statement of Theorem 3.2 can also be formulated for repellers of continuous expanding maps (see [13] for the definition).

We can formulate much stronger statements for **conformal** expanding maps.

THEOREM 3.3: *If J is a repeller of a topologically mixing $C^{1+\varepsilon}$ expanding map f, for some $\varepsilon > 0$, and f is conformal on J, then the following properties hold:*

1. $h(f|\mathfrak{B}_f) = h(f|J)$ *and* $\dim_H \mathfrak{B}_f = \dim_H J$;
2. $m_D \neq m_E$ *if and only if* $h(f|\mathfrak{L}_f) = h(f|J)$ *and* $\dim_H \mathfrak{L}_f = \dim_H J$.

If in addition $\mu \in G(f|J)$ *then*

3. $\mu \neq m_D$ *if and only if* $h(f|\mathfrak{D}(\mu)) = h(f|J)$ *and* $\dim_H \mathfrak{D}(\mu) = \dim_H J$;
4. $\mu \neq m_E$ *if and only if* $h(f|\mathfrak{H}_f(\mu)) = h(f|J)$ *and* $\dim_H \mathfrak{H}_f(\mu) = \dim_H J$;
5. *the three measures* μ, m_D, *and* m_E *are distinct if and only if*

$$h(f|\mathfrak{D}(\mu) \cap \mathfrak{H}_f(\mu) \cap \mathfrak{L}_f) = h(f|J)$$

and

$$\dim_H(\mathfrak{D}(\mu) \cap \mathfrak{H}_f(\mu) \cap \mathfrak{L}_f) = \dim_H J.$$

We note that all the identities concerning the topological entropy are immediate consequences of Theorem 2.1. Theorem 3.3 follows from a much more general statement in Theorem 7.1.

4. Horseshoes

4.1. DESCRIPTION OF THE RESULTS. Let $f: M \to M$ be a C^1 diffeomorphism of a smooth manifold, and $\Lambda \subset M$ a compact locally maximal hyperbolic set of f. Then, there is a continuous splitting of the tangent bundle $T_\Lambda M = E^s \oplus E^u$, and constants $C > 0$ and $\lambda \in (0, 1)$ such that for each $x \in \Lambda$:

1. $d_x f\, E^s_x = E^s_{fx}$ and $d_x f\, E^u_x = E^u_{fx}$;
2. $\|d_x f^n v\| \leq C\lambda^n \|v\|$ for all $v \in E^s_x$ and $n \geq 0$;
3. $\|d_x f^{-n} v\| \leq C\lambda^n \|v\|$ for all $v \in E^u_x$ and $n \geq 0$.

For each point $x \in \Lambda$ there exist **local stable** and **unstable manifolds** $W^s(x)$ and $W^u(x)$, with $T_xW^s(x) = E^s_x$ and $T_xW^u(x) = E^u_x$. Moreover, there exists $\delta > 0$ such that for all $x, y \in \Lambda$ with $\rho(x,y) < \delta$, the set $W^s(x) \cap W^u(y)$ consists of a single point, which we denote by $[x,y]$, and the map

$$[\cdot,\cdot]: \{(x,y) \in \Lambda \times \Lambda: \rho(x,y) < \delta\} \to M$$

is continuous. For each $x \in M$, we write

$$a^u(x) = \|d_xf|E^u(x)\| \quad \text{and} \quad a^s(x) = \|d_xf|E^s(x)\|.$$

The functions a^s and a^u satisfy $a^u(x) > 1$ and $a^s(x) < 1$ for every $x \in \Lambda$, and they are Hölder continuous if f is of class $C^{1+\varepsilon}$.

Locally maximal hyperbolic sets have Markov partitions of arbitrarily small diameter. Each Markov partition has associated a two-sided subshift of finite type $\sigma|\Sigma$, and a **coding map** $\chi: \Sigma \to \Lambda$ for the hyperbolic set, which is Hölder continuous, onto, and satisfies $f \circ \chi = \chi \circ \sigma$ and $\sup\{\mathrm{card}(\chi^{-1}x): x \in \Lambda\} < \infty$ (see, for example, [13] for details). For each point $\omega = (\cdots i_{-1}i_0i_1\cdots) \in \Sigma$, and each non-negative integers n, m, we define the cylinder

$$C^n_m = C^n_m(\omega) = \{(\cdots j_{-1}j_0j_1\cdots) \in \Sigma: j_k = i_k \text{ for } i = -m, \dots, n\}.$$

We now describe several irregular sets which carry full topological entropy and full Hausdorff dimension.

THEOREM 4.1: *If Λ is a compact locally maximal hyperbolic set of a topologically mixing C^1 diffeomorphism f, then the following properties hold:*

1. $h(f|\mathfrak{B}_f) = h(f|\Lambda)$;
2. *if $\mu \in G(f|\Lambda)$ and $\mu \neq m_E$, then $\mathfrak{H}_f(\mu) \subset \mathfrak{B}_f$ and $h(f|\mathfrak{H}_f(\mu)) = h(f|\Lambda)$.*

The statement in Theorem 4.1 can also be formulated for basic sets of Axiom A$^\sharp$ homeomorphisms (see [13] for the definition).

We can formulate much stronger statements for **surface** diffeomorphisms, and, more generally, for diffeomorphisms on manifolds of arbitrary dimension such that f is **conformal** on Λ, i.e., such that $d_xf|E^u(x)$ and $d_xf|E^s(x)$ are multiples of isometries for each $x \in \Lambda$.

Let $\mathfrak{M}_D$ be the set of f-invariant measures μ such that $\dim_H \mu = \dim_H \Lambda$. Note that $\mathfrak{M}_D$ may be empty. Let Λ be a locally maximal hyperbolic set of the C^1 diffeomorphism f on a compact surface. We denote by d^u and d^s the unique roots of the equations

$$P_\Lambda(-d^u \log a^u) = 0 \quad \text{and} \quad P_\Lambda(d^s \log a^s) = 0,$$

where $P_\Lambda(g)$ denotes the topological pressure of g with respect to f on the set Λ. In [12], McCluskey and Manning proved that

$$\dim_H(W^u(x)\cap\Lambda)=d^u \quad \text{and} \quad \dim_H(W^s(x)\cap\Lambda)=d^s$$

for every $x\in\Lambda$; moreover $\dim_H\Lambda=d^u+d^s$. They also showed that $\mathfrak{M}_D\neq\emptyset$ if and only if the functions $-d^u\log a^u$ and $d^s\log a^s$ are cohomologous; in this case the equilibrium measure of $-d^u\log a^u$ and $d^s\log a^s$ is the unique probability measure belonging to $\mathfrak{M}_D$. We have $\mathfrak{M}_D=\emptyset$ for diffeomorphisms in a C^2 open dense set, with the dimension of any invariant measure uniformly bounded away from the Hausdorff dimension of the horseshoe: $\sup_\mu\dim_H\mu<\dim_H\Lambda$, where the supremum is taken over all f-invariant Borel probability measures in Λ.

THEOREM 4.2: *If f is a topologically mixing $C^{1+\varepsilon}$ diffeomorphism, for some $\varepsilon>0$, and f is conformal on a compact locally maximal saddle-type hyperbolic set Λ of f, then the following properties hold:*

1. $h(f|\mathfrak{B}_f)=h(f|\Lambda)$ *and* $\dim_H\mathfrak{B}_f=\dim_H\Lambda$;
2. $\log a^u$ *is non-cohomologous to* 0 *if and only if* $h(f|\mathfrak{L}_f)=h(f|\Lambda)$ *and* $\dim_H\mathfrak{L}_f=\dim_H\Lambda$;
3. $\log a^s$ *is non-cohomologous to* 0 *if and only if* $h(f|\mathfrak{L}_{f^{-1}})=h(f|\Lambda)$ *and* $\dim_H\mathfrak{L}_{f^{-1}}=\dim_H\Lambda$.

If in addition $\mu\in G(f|\Lambda)$ then

4. $\mu\notin\mathfrak{M}_D$ *if and only if* $h(f|\mathfrak{D}(\mu))=h(f|\Lambda)$ *and* $\dim_H\mathfrak{D}(\mu)=\dim_H\Lambda$;
5. $\mu\neq m_E$ *if and only if* $h(f|\mathfrak{H}_f(\mu))=h(f|\Lambda)$ *and* $\dim_H\mathfrak{H}_f(\mu)=\dim_H\Lambda$;
6. $\mu\neq m_E$ *and* $\mu\notin\mathfrak{M}_D$ *if and only if* $h(f|\mathfrak{D}(\mu)\cap\mathfrak{H}_f(\mu))=h(f|\Lambda)$ *and* $\dim_H(\mathfrak{D}(\mu)\cap\mathfrak{H}_f(\mu))=\dim_H\Lambda$.

In [10], Eckmann and Ruelle discussed the pointwise dimension of hyperbolic measures μ (that is, measures with non-zero Lyapunov exponents almost everywhere), invariant under diffeomorphisms. They conjectured that $\mu(\mathfrak{D}(\mu))=0$. This claim has been known as the Eckmann–Ruelle conjecture and has become a celebrated problem in the dimension theory of dynamical systems. In [3], Barreira, Pesin, and Schmeling establish the affirmative solution of this conjecture for $C^{1+\varepsilon}$ diffeomorphisms (an announcement appeared in [2]).

It was established by Shereshevsky in [19] that $\dim_H\mathfrak{D}(\mu)>0$, and $\overline{\mathfrak{D}(\mu)}\supset\Lambda$ for a generic C^2 surface diffeomorphism possessing a locally maximal hyperbolic set Λ, and a generic Hölder continuous potential, with respect to the C^0 topology, with Gibbs measure μ. By Theorem 2 of McCluskey and Manning in [12], this is an immediate consequence of Theorem 2.1, and Statement 4 in Theorem 4.2.

Let $\mathfrak{G}(\nu)$ be the set of generic points for the measure ν (see Section 2.3). The following is a simple consequence of Theorem 4.2, and Proposition 6.5 below.

THEOREM 4.3: *For a surface diffeomorphism f in a C^2 open dense set, possessing a compact locally maximal hyperbolic set Λ, if $\mu \in G(f|\Lambda)$ and $\mu \neq m_E$, then*

$$\dim_H(\mathfrak{D}(\mu) \cap \mathfrak{H}_f(\mu) \cap \mathfrak{L}_f \cap \mathfrak{L}_{f^{-1}}) > \dim_H \bigcup_\nu \mathfrak{G}(\nu).$$

In [11], Katok proved that for an ergodic hyperbolic measure μ (i.e., an ergodic measure with non-zero Lyapunov exponents), invariant under a $C^{1+\varepsilon}$ diffeomorphism $f\colon M \to M$, given $\delta > 0$ there exists a closed f-invariant hyperbolic set $\Gamma \subset M$ such that the restriction of f to Γ is topologically conjugate to a subshift of finite type with topological entropy $h(f|\Gamma) \geq h_\mu(f) - \delta$. In other words, the entropy of a hyperbolic measure can be approximated by the topological entropies of invariant hyperbolic sets. If μ is a hyperbolic measure we denote by μ_x^s and μ_x^u the conditional measures on the families of local stable and unstable manifolds. Using this approximation result we obtain the following.

THEOREM 4.4: *Let f be a topologically mixing $C^{1+\varepsilon}$ diffeomorphism of a compact manifold M, for some $\varepsilon > 0$, and μ an f-invariant hyperbolic measure whose support is the whole manifold. If $\mathfrak{L}_f \neq \emptyset$, then the following properties hold:*

1. $h(f|\mathfrak{L}_f) \geq h_\mu(f)$;
2. *if M is a surface, then* $\dim_H \mathfrak{L}_f \geq \dim_H \mu_x^s + \dim_H \mu_x^u$.

4.2. ENTROPY FOR INVERTIBLE TRANSFORMATIONS. When we change from a one-sided to a two-sided shift (coding a hyperbolic set), there is an asymmetry which apparently was never mentioned in the literature. This problem occurs only for non-compact or non-invariant subsets of Σ. We note that the irregular sets in which we are interested are always invariant but are never compact; in fact they are everywhere dense sets of zero measure.

Let X be a compact metric space, and $f\colon X \to X$ a homeomorphism. The problem alluded to above is that, with the definition of topological entropy introduced in [7] (see also [14]), $h(f|Z)$ and $h(f^{-1}|Z)$ may not coincide for an arbitrary set $Z \subset X$; however if Z is compact and f-invariant, then $h(f|Z) = h(f^{-1}|Z)$.

We introduce a new notion of topological entropy which takes into account the "complexity" both in the "future" and in the "past". We assume that $f\colon X \to X$ is continuous but not necessarily invertible. For each finite cover $\mathfrak{U}$ of X, we denote by $\mathfrak{W}_n(\mathfrak{U})$ the collection of strings $\mathbf{U} = U_0 \cdots U_n$ of sets $U_0, \ldots, U_n \in \mathfrak{U}$. For each $\mathbf{U} \in \mathfrak{W}_n(\mathfrak{U})$, we call the integer $m(\mathbf{U}) = n$ the **length** of $\mathbf{U}$, and define the open set

$$X(\mathbf{U}) = \{x \in X\colon f^k x \in U_k \text{ for } k = 0, \dots, n\}.$$

For every set $Z \subset X$ and every real number α, we set

$$N(Z, \alpha, \mathfrak{U}) = \lim_{n\to\infty} \inf_{\Gamma} \sum_{(\mathbf{U},\mathbf{V})\in\Gamma} \exp[-\alpha m(\mathbf{U}) - \alpha m(\mathbf{V})], \tag{8}$$

where the infimum is taken over all finite or countable collections

$$\Gamma \subset \coprod_{k+\ell>n} \mathfrak{W}_k(\mathfrak{U}) \times \mathfrak{W}_\ell(\mathfrak{U})$$

such that

$$\bigcup_{(\mathbf{U},\mathbf{V})\in\Gamma} X(\mathbf{U}) \cap f^{m(\mathbf{V})} X(\mathbf{V}) \supset Z.$$

By a simple modification of the construction of Carathéodory dimension characteristics (see [13]), when α goes from $-\infty$ to $+\infty$, the quantity in (8) jumps from $+\infty$ to 0 at a unique critical value. Hence, we can define the number

$$h^*(f|Z, \mathfrak{U}) = \inf\{\alpha\colon N(Z, \alpha, \mathfrak{U}) = 0\} = \sup\{\alpha\colon N(Z, \alpha, \mathfrak{U}) = +\infty\}.$$

One can show that the following limit exists (compare with the proof of Theorem 6.1 below):

$$h^*(f|Z) = \lim_{\operatorname{diam}\mathfrak{U}\to 0} h^*(f|Z, \mathfrak{U}).$$

We call $h^*(f|Z)$ the **two-sided topological entropy** of f on the set Z. We have $h^*(f|Z) \le h(f|Z)$ and this inequality may be strict. For example, if Z is a local unstable manifold for an Anosov diffeomorphism f, then $0 = h^*(f|Z) < h(f|Z)$.

When f is a homeomorphism, one can show that for every subset $Z \subset \Sigma$, we have

$$h^*(f|Z) \le \min\{h(f|Z), h(f^{-1}|Z)\}, \tag{9}$$

the minimum of the contributions from the "future" and from the "past", respectively; moreover, $h^*(f|Z) = h^*(f^{-1}|Z)$. For example, if Z is the union of a local stable manifold and a local unstable manifold for an Anosov diffeomorphism f, then $h^*(f|Z) = 0$ and (9) is a strict inequality.

Let Λ be a locally maximal hyperbolic set of a diffeomorphism f. Clearly, $h^*(f|\Lambda) = h(f|\Lambda)$. In Theorems 4.1 and 4.2 we described f-invariant noncompact sets of zero measure that carry full topological entropy and full Hausdorff dimension. One can replace h by h^* in every statement of Theorems 4.1 and 4.2. In particular, one can prove the following.

THEOREM 4.5: *Let f be a topologically mixing $C^{1+\varepsilon}$ diffeomorphism, for some $\varepsilon > 0$, such that f is conformal on a compact locally maximal saddle-type hyperbolic set Λ of f. If $\mu \in G(f|\Lambda)$, $\mu \neq m_E$, and $\mu \notin \mathfrak{M}_D$, then $h^*(f|\mathfrak{D}(\mu) \cap \mathfrak{H}_f(\mu)) = h^*(f|\Lambda)$.*

5. Irregular parts of multifractal spectra

The irregular sets defined in Sections 2, 3, and 4 are "closely" related to the irregular parts of multifractal spectra (see [4, 5]). Let X be a complete separable metric space, and $g\colon Y \to [-\infty, +\infty]$ a function defined on a subset $Y \subset X$. The level sets of g,

$$K_\alpha^g = \{x \in X\colon g(x) = \alpha\},$$

for $-\infty \leq \alpha \leq +\infty$, are disjoint and produce a **multifractal decomposition** of X, that is,

$$X = \bigcup_{-\infty \leq \alpha \leq +\infty} K_\alpha^g \cup (X \setminus Y).$$

Let now G be a real function defined on the collection of subsets of X. Assume that $G(Z_1) \leq G(Z_2)$ if $Z_1 \subset Z_2$. We define the function $\mathcal{F}\colon [-\infty, +\infty] \to \mathbb{R}$ by

$$\mathcal{F}(\alpha) = G(K_\alpha^g).$$

We call $\mathcal{F}$ the **multifractal spectrum** specified by the pair of functions (g, G). The set Y is called the **irregular part** of $\mathcal{F}$ (or simply of g), and is denoted by $\mathcal{I}_\mathcal{F} = \mathcal{I}_g$.

Let $f\colon X \to X$ be a continuous map, and μ a Borel probability measure on X. We consider two set functions on X. Namely, given a subset $Z \subset X$, let

$$G_D(Z) = \dim_H Z \quad \text{and} \quad G_E(Z) = h(f|Z).$$

Consider the subset $Y_D \subset X$ consisting of all points $x \in X$ for which there exists the limit

$$d_\mu(x) = \lim_{r \to 0} \frac{\log \mu(B(x,r))}{\log r}.$$

The number $d_\mu(x)$ is called the **pointwise dimension** of μ at x. We obtain the multifractal spectra specified by the pairs (d_μ, G_D) and (d_μ, G_E). We have $\mathcal{I}_{d_\mu} = X \setminus Y_D$.

Assume, in addition, that f preserves μ. Consider a finite measurable partition ξ of X, and the set $Y_{E,\xi} \subset X$ consisting of all points $x \in X$ for which there exists the limit

$$h_\mu(x) = h_\mu(f, \xi, x) = \lim_{n \to \infty} -\frac{1}{n} \log \mu(\xi_n(x)).$$

We call $h_\mu(f,\xi,x)$ the μ-**local entropy** of f at the point x (with respect to ξ). Clearly, $Y_{E,\xi}$ is f-invariant and $h_\mu(f,\xi,fx) = h_\mu(f,\xi,x)$ for every $x \in Y_{E,\xi}$. By the Shannon–McMillan–Breiman theorem, $\mu(Y_{E,\xi}) = 1$. If ξ is a generating partition and μ is ergodic, then $h_\mu(f) = h_\mu(f,\xi,x)$ for μ-almost all $x \in X$. We obtain the multifractal spectra specified by the pairs (h_μ, G_D) and (h_μ, G_E). In some situations these spectra do not depend on ξ for a broad class of partitions (see [4, 5]). We have $\mathcal{I}_{h_\mu} = X \setminus Y_{E,\xi}$.

Let now X be a differentiable manifold and $f\colon X \to X$ a C^1 map. Consider the subset $Y_L \subset X$ of all points $x \in X$ for which there exists the limit

$$\chi(x) = \lim_{n\to+\infty} \frac{1}{n} \log \|d_x f^n\|.$$

By the Kingman's subadditive ergodic theorem, if μ is an f-invariant Borel probability measure then $\mu(Y_L) = 1$. We obtain two multifractal spectra specified respectively by the pairs of functions (χ, G_D) and (χ, G_E). We have $\mathcal{I}_\chi = X \setminus Y_L$.

Using the same definitions and notations of Sections 2, 3, and 4, we now reformulate the statements in those sections for irregular parts of multifractal spectra.

THEOREM 5.1: *Let $\sigma|\Sigma$ be a topologically mixing subshift of finite type. The Hölder continuous functions $g_1, \dots, g_k$ on Σ are non-cohomologous to 0 if and only if $\mathcal{I}_{g_1} \cap \dots \cap \mathcal{I}_{g_k}$ is a proper dense subset and $h(\sigma|\mathcal{I}_{g_1} \cap \dots \cap \mathcal{I}_{g_k}) = h(\sigma)$.*

THEOREM 5.2: *Let J be a repeller of a topologically mixing $C^{1+\varepsilon}$ expanding map f, for some $\varepsilon > 0$, such that f is conformal on J, and $\mu \in G(f|J)$. The three measures μ, m_D, and m_E are distinct if and only if $\mathcal{I}_{d_\mu} \cap \mathcal{I}_{h_\mu} \cap \mathcal{I}_\chi$ is a proper dense subset, and*

$$h(f|\mathcal{I}_{d_\mu} \cap \mathcal{I}_{h_\mu} \cap \mathcal{I}_\chi) = h(f|J)$$

and

$$\dim_H(\mathcal{I}_{d_\mu} \cap \mathcal{I}_{h_\mu} \cap \mathcal{I}_\chi) = \dim_H J.$$

THEOREM 5.3: *Let f be a topologically mixing $C^{1+\varepsilon}$ surface diffeomorphism, for some $\varepsilon > 0$, Λ a compact locally maximal saddle-type hyperbolic set of f, and $\mu \in G(f|\Lambda)$. We have $\mu \neq m_E$ and $\mu \notin \mathfrak{M}_D$ if and only if $\mathcal{I}_{d_\mu} \cap \mathcal{I}_{h_\mu}$ is a proper dense subset, and*

$$h(f|\mathcal{I}_{d_\mu} \cap \mathcal{I}_{h_\mu}) = h(f|\Lambda) \quad \text{and} \quad \dim_H(\mathcal{I}_{d_\mu} \cap \mathcal{I}_{h_\mu}) = \dim_H \Lambda.$$

The irregular parts of multifractal spectra can naturally be viewed as irregular sets. Our approach to prove that irregular sets carry full topological entropy and full Hausdorff dimension exploits this relationship, and, to some extent, it shows that it is enough to deal with irregular parts of multifractal spectra.

6. A new Carathéodory dimension characteristic

6.1. DESCRIPTION AND MAIN PROPERTIES OF A NEW CARATHÉODORY DIMENSION CHARACTERISTIC. Let X be a compact metric space, and $f\colon X \to X$ a continuous map (it need not be invertible). We use the notation of Section 4.2, and say that the collection of strings $\Gamma \subset \bigcup_{n\geq 1} \mathfrak{W}_n(\mathfrak{U})$ **covers** the set $Z \subset X$ if $\bigcup_{\mathbf{U}\in\Gamma} X(\mathbf{U}) \supset Z$.

Let $u\colon X \to \mathbb{R}$ be a strictly positive continuous function. For each string $\mathbf{U} \in \mathfrak{W}_n(\mathfrak{U})$, we write

$$u(\mathbf{U}) = \begin{cases} \sup\left\{\sum_{k=0}^{m(\mathbf{U})} u(f^k x)\colon x \in X(\mathbf{U})\right\} & \text{if } X(\mathbf{U}) \neq \emptyset, \\ -\infty & \text{if } X(\mathbf{U}) = \emptyset. \end{cases}$$

For each set $Z \subset X$ and each real number α, we define

$$M(Z,\alpha,u,\mathfrak{U}) = \lim_{n\to\infty} \inf_{\Gamma} \sum_{\mathbf{U}\in\Gamma} \exp(-\alpha u(\mathbf{U})), \tag{10}$$

where the infimum is taken over all finite or countable collections

$$\Gamma \subset \bigcup_{k\geq n} \mathfrak{W}_k(\mathfrak{U})$$

that cover Z. Likewise, we define

$$\underline{M}(Z,\alpha,u,\mathfrak{U}) = \liminf_{n\to\infty} \inf_{\Gamma} \sum_{\mathbf{U}\in\Gamma} \exp(-\alpha u(\mathbf{U})), \tag{11}$$

$$\overline{M}(Z,\alpha,u,\mathfrak{U}) = \limsup_{n\to\infty} \inf_{\Gamma} \sum_{\mathbf{U}\in\Gamma} \exp(-\alpha u(\mathbf{U})), \tag{12}$$

where the infimum is now taken over all finite or countable collections $\Gamma \subset \mathfrak{W}_n(\mathfrak{U})$ that cover Z.

By a slight modification of the construction of Carathéodory dimension characteristics (see [13]), when α goes from $-\infty$ to $+\infty$, each of the quantities in (10), (11), and (12) jumps from $+\infty$ to 0 at a unique critical value. Hence, we can define the numbers

$$\begin{aligned}\dim_{u,\mathfrak{U}} Z &= \inf\{\alpha\colon M(Z,\alpha,u,\mathfrak{U})=0\}\\ &= \sup\{\alpha\colon M(Z,\alpha,u,\mathfrak{U})=+\infty\},\\ \underline{\dim}_{u,\mathfrak{U}} Z &= \inf\{\alpha\colon \underline{M}(Z,\alpha,u,\mathfrak{U})=0\}\\ &= \sup\{\alpha\colon \underline{M}(Z,\alpha,u,\mathfrak{U})=+\infty\},\\ \overline{\dim}_{u,\mathfrak{U}} Z &= \inf\{\alpha\colon \overline{M}(Z,\alpha,u,\mathfrak{U})=0\}\\ &= \sup\{\alpha\colon \overline{M}(Z,\alpha,u,\mathfrak{U})=+\infty\}.\end{aligned}$$

THEOREM 6.1: *The following limits exist:*

$$\dim_u Z \stackrel{\text{def}}{=} \lim_{\operatorname{diam}\mathfrak{U}\to 0} \dim_{u,\mathfrak{U}} Z,$$

$$\underline{\dim}_u Z \stackrel{\text{def}}{=} \lim_{\operatorname{diam}\mathfrak{U}\to 0} \underline{\dim}_{u,\mathfrak{U}} Z, \quad \overline{\dim}_u Z \stackrel{\text{def}}{=} \lim_{\operatorname{diam}\mathfrak{U}\to 0} \overline{\dim}_{u,\mathfrak{U}} Z.$$

We call $\dim_u Z$ the **u-dimension** of Z, and $\underline{\dim}_u Z$ and $\overline{\dim}_u Z$ the **lower** and **upper u-capacities** of Z (specified by the map f). We note that Z need not be compact nor f-invariant. The following is an immediate consequence of the general theory of Carathéodory dimension characteristics (see [13]).

THEOREM 6.2: *The following properties hold:*

1. $0 \le \dim_u Z \le \underline{\dim}_u Z \le \overline{\dim}_u Z$;
2. $\dim_u \emptyset = \underline{\dim}_u \emptyset = \overline{\dim}_u \emptyset = 0$;
3. *if* $Z_1 \subset Z_2$, *then* $\dim_u Z_1 \le \dim_u Z_2$, $\underline{\dim}_u Z_1 \le \underline{\dim}_u Z_2$, *and* $\overline{\dim}_u Z_1 \le \overline{\dim}_u Z_2$;
4. *if* $Z = \bigcup_{i\in I} Z_i$ *is a union of sets* $Z_i \subset X$, *with* I *at most countable, then:*
   (a) $\dim_u Z = \sup_{i\in I} \dim_u Z_i$;
   (b) $\underline{\dim}_u Z \ge \sup_{i\in I} \underline{\dim}_u Z_i$, *with equality if* I *is finite, and* $\underline{\dim}_u Z_i = \overline{\dim}_u Z_i$ *for each* $i \in I$;
   (c) $\overline{\dim}_u Z \ge \sup_{i\in I} \overline{\dim}_u Z_i$, *with equality if* I *is finite;*
5. *if* $h\colon X \to X$ *is a homeomorphism such that* $f \circ h = h \circ f$, *then* $\dim_u Z = \dim_{u\circ h^{-1}} h(Z)$, $\underline{\dim}_u Z = \underline{\dim}_{u\circ h^{-1}} h(Z)$, *and* $\overline{\dim}_u Z = \overline{\dim}_{u\circ h^{-1}} h(Z)$;
6. *if* u, $v\colon X \to \mathbb{R}$ *are strictly positive continuous functions, then* $|\dim_u Z - \dim_v Z| \le \|u-v\|$, $|\underline{\dim}_u Z - \underline{\dim}_v Z| \le \|u-v\|$, *and* $|\overline{\dim}_u Z - \overline{\dim}_v Z| \le \|u-v\|$.

Examples: 1. If $u \equiv 1$, then for each set $Z \subset X$, the number $\dim_u Z$ coincides with the topological entropy of f on Z, and the numbers $\underline{\dim}_u Z$ and $\overline{\dim}_u Z$ coincide, respectively, with the lower and upper capacity topological entropies of f on Z (note that the set Z need not be compact or f-invariant; see [13] for the definitions).

2. If $u = \log a$ where a is the norm of the derivative of a conformal expanding map with repeller Z (see Section 3), then the number $\dim_u Z$ coincides with $\dim_H Z$, and the numbers $\underline{\dim}_u Z$ and $\overline{\dim}_u Z$ coincide, respectively, with the lower and upper box dimensions of Z. This follows immediately from the existence of universal constants c_1, $c_2 > 0$ such that $c_1(\operatorname{diam} X(\mathbf{U}))^\alpha \leq \exp(-\alpha u(\mathbf{U})) \leq c_2(\operatorname{diam} X(\mathbf{U}))^\alpha$.

We now follow the approach of Pesin in [13] to define Carathéodory characteristics of measures. For every Borel probability measure μ on X (it need not be f-invariant), we set

$$\dim_{u,\mathfrak{U}} \mu = \inf\{\dim_{u,\mathfrak{U}} Z \colon \mu(Z) = 1\},$$

$$\underline{\dim}_{u,\mathfrak{U}}\mu = \lim_{\delta\to 0} \inf\{\underline{\dim}_{u,\mathfrak{U}} Z \colon \mu(Z) \geq 1-\delta\},$$

$$\overline{\dim}_{u,\mathfrak{U}}\mu = \lim_{\delta\to 0} \inf\{\overline{\dim}_{u,\mathfrak{U}} Z \colon \mu(Z) \geq 1-\delta\}.$$

It follows from Theorem 6.1 that there exist the limits

$$\dim_u \mu \stackrel{\text{def}}{=} \lim_{\operatorname{diam}\mathfrak{U}\to 0} \dim_{u,\mathfrak{U}} \mu,$$

$$\underline{\dim}_u\mu \stackrel{\text{def}}{=} \lim_{\operatorname{diam}\mathfrak{U}\to 0} \underline{\dim}_{u,\mathfrak{U}}\mu, \quad \overline{\dim}_u\mu \stackrel{\text{def}}{=} \lim_{\operatorname{diam}\mathfrak{U}\to 0} \overline{\dim}_{u,\mathfrak{U}}\mu.$$

We call $\dim_u \mu$ the **u-dimension** of μ, and $\underline{\dim}_u\mu$ and $\overline{\dim}_u\mu$ the **lower** and **upper u-capacities** of μ (specified by the map f).

Example: If $u \equiv 1$, then the number $\dim_u \mu$ coincides with the μ-metric entropy of f, and the numbers $\underline{\dim}_u\mu$ and $\overline{\dim}_u\mu$ coincide, respectively, with the lower and upper μ-metric capacity entropies of f (see [13] for the definition).

We define the **lower** and **upper u-pointwise dimensions** of μ at the point $x \in X$ by

$$\underline{d}_{\mu,u}(x,\mathfrak{U}) = \liminf_{n\to\infty} \inf -\frac{\log \mu(X(\mathbf{U}))}{u(\mathbf{U})}$$

and

$$\overline{d}_{\mu,u}(x,\mathfrak{U}) = \limsup_{n\to\infty} \sup -\frac{\log \mu(X(\mathbf{U}))}{u(\mathbf{U})},$$

where the infimum and supremum are taken over all strings $\mathbf{U} \in \mathfrak{W}_n(\mathfrak{U})$ such that $x \in X(\mathbf{U})$.

Let ξ be a partition of X. For each $n \in \mathbb{N}$, we define a new partition of X by $\xi_n = \xi \vee f^{-1}\xi \vee \cdots \vee f^{-n}\xi$, and denote by $\xi_n(x)$ the atom of ξ_n containing the point $x \in X$. We denote by $h_\mu(f)$ the μ-measure-theoretic entropy of f. For each function u on X, we write $S_n u(x) = \sum_{k=0}^n u(f^k x)$ for each $x \in X$ and each $n \in \mathbb{N}$. The following is an immediate consequence of the Birkhoff Ergodic Theorem, the Shannon–McMillan–Breiman Theorem, and Theorem 4.1 in [13].

THEOREM 6.3: *If μ is an ergodic f-invariant Borel probability measure on X, then:*

1. *if ξ is a generating partition of X, then, for μ-almost every $x \in X$,*

$$\begin{aligned}\lim_{\operatorname{diam}\mathfrak{U}\to 0} \underline{d}_{\mu,u}(x,\mathfrak{U}) &= \lim_{\operatorname{diam}\mathfrak{U}\to 0} \overline{d}_{\mu,u}(x,\mathfrak{U}) \\ &= \lim_{n\to\infty} -\frac{\log\mu(\xi_n(x))}{S_n u(x)} = \frac{h_\mu(f)}{\int_X u\,d\mu} \stackrel{\text{def}}{=} d;\end{aligned}$$

2. $\dim_u \mu = \underline{\dim}_u \mu = \overline{\dim}_u \mu = d$.

The following result expresses a relation between the u-dimension and the topological pressure. Let $g\colon X \to X$ be a continuous function. For each real number β, we set

$$p(Z,\beta,\mathfrak{U}) = \lim_{n\to\infty} \inf_{\Gamma} \sum_{\mathbf{U}\in\Gamma} \exp(-\beta m(\mathbf{U}) + g(\mathbf{U})),$$

where the infimum is taken over all finite or countable collections

$$\Gamma \subset \bigcup_{k\geq n} \mathfrak{W}_k(\mathfrak{U})$$

that cover Z. As β runs from $-\infty$ to $+\infty$, the number $p(Z,\beta,\mathfrak{U})$ jumps from $+\infty$ to 0 at a unique critical value denoted $P_Z(g,\mathfrak{U})$. Moreover, the limit $P_Z(g) = \lim_{\operatorname{diam}\mathfrak{U}\to 0} P_Z(g,\mathfrak{U})$ exists and is called the **topological entropy of g (on the set Z)**. Set $g = -\alpha u$. Then $p(Z,0,\mathfrak{U}) = M(Z,\alpha,u,\mathfrak{U})$ and we obtain the following result.

PROPOSITION 6.4 (Bowen pressure formula): *We have $\dim_u Z = \alpha$, where α is the unique root of the equation $P_Z(-\alpha u) = 0$.*

The Bowen pressure formula was introduced by Bowen in [9] in the context of quasi-circles. See [1] for additional references.

One can easily obtain similar statements to that of Proposition 6.4 associating the lower and upper u-dimension, respectively, and the lower and upper capacity topological pressures (see [13] for the definition).

We now present a variational principle for the u-dimension.

PROPOSITION 6.5: *We have*

$$\dim_u \bigcup_\mu \mathfrak{G}(\mu) = \sup\{\dim_u \mu \colon \mu \in \mathfrak{M} \text{ is ergodic}\}. \tag{13}$$

We note that the union in (13) is in general not countable; otherwise, Proposition 6.5 would follow immediately from Statement 4a in Theorem 6.2.

6.1. COMPLETE MULTIFRACTAL ANALYSIS FOR THE NEW CARATÉODORY DIMENSION. We will present a complete multifractal analysis for the u-dimension, specified by a subshift of finite type $\sigma|\Sigma$. Let μ be a Borel probability measure on Σ. For every $x \in \Sigma$, we write

$$\underline{d}_{\mu,u}(x) = \liminf_{n\to\infty} -\frac{\log \mu(C_n(x))}{S_n u(x)}$$

and

$$\overline{d}_{\mu,u}(x) = \limsup_{n\to\infty} -\frac{\log \mu(C_n(x))}{S_n u(x)}.$$

One can easily show that if u is Hölder continuous, then

$$\underline{d}_{\mu,u}(x) = \underline{d}_{\mu,u}(x,\mathfrak{U}) \quad \text{and} \quad \overline{d}_{\mu,u}(x) = \overline{d}_{\mu,u}(x,\mathfrak{U})$$

for every $x \in \Sigma$ and open cover $\mathfrak{U}$ of Σ by cylinders (not necessarily all with the same length).

For every real number α, set

$$K_\alpha = \{x \in \Sigma \colon \underline{d}_{\mu,u}(x) = \overline{d}_{\mu,u}(x) = \alpha\}.$$

Whenever $K_\alpha \neq \emptyset$ and $x \in K_\alpha$, we denote the common value α of $\underline{d}_{\mu,u}(x)$ and $\overline{d}_{\mu,u}(x)$ by $d_{\mu,u}(x)$, and call it the **u-pointwise dimension of μ at x**. We set

$$\mathfrak{D}_u(\alpha) = \dim_u K_\alpha.$$

The function $\alpha \mapsto \mathfrak{D}_u(\alpha)$ is called the **u-dimension spectrum for u-pointwise dimensions** (with respect to the measure μ). Let ϕ be a continuous function on Σ. For every real number q, we define the function

$$\phi_q = -T_u(q)u + q\phi,$$

where the number $T_u(q)$ is chosen such that $P(\phi_q) = 0$. We denote by ν_q and m_u, respectively, the equilibrium measures of ϕ_q and $-\dim_u \Sigma \cdot u$ with respect to σ.

The following is a complete multifractal analysis of the spectrum $\mathfrak{D}_u$ for subshifts of finite type.

THEOREM 6.6: *Let $\sigma|\Sigma$ be a one-sided or two-sided topologically mixing subshift of finite type, u and ϕ Hölder continuous functions on Σ, such that u is positive and $P(\phi) = 0$, and μ the equilibrium measure of ϕ with respect to σ. Then, the following properties hold:*

1. *For μ-almost every $x \in \Sigma$, the u-pointwise dimension of μ at x exists and*
$$d_{\mu,u}(x) = -\frac{\int_\Sigma \phi\, d\mu}{\int_\Sigma u\, d\mu} = \frac{h_\mu(\sigma)}{\int_\Sigma u\, d\mu}.$$
2. *The function $q \mapsto T_u(q)$ is real analytic on $\mathbb{R}$, and satisfies $T_u'(q) \le 0$ and $T_u''(q) \ge 0$ for every $q \in \mathbb{R}$. Moreover, $T_u(0) = \dim_u \Sigma$ and $T_u(1) = 0$.*
3. *The domain of the function $\alpha \mapsto \mathfrak{D}_u(\alpha)$ is a closed interval in $[0, +\infty)$ and coincides with the range of the function $\alpha_u(q) = -T_u'(q)$. For every $q \in \mathbb{R}$, we have*
$$\mathfrak{D}_u(\alpha_u(q)) = T_u(q) + q\alpha_u(q),$$
and
$$\alpha_u(q) = -\frac{\int_\Sigma \phi\, d\nu_q}{\int_\Sigma u\, d\nu_q}.$$
4. *For every $q \in \mathbb{R}$, $\nu_q(K_{\alpha_u(q)}) = 1$, and*
$$d_{\nu_q,u}(x) = T_u(q) + q\alpha_u(q)$$
for ν_q-almost all $x \in K_{\alpha_u(q)}$. Moreover, $\overline{d}_{\nu_q,u}(x) \le T_u(q) + q\alpha_u(q)$ for every $x \in K_{\alpha_u(q)}$, and $\mathfrak{D}_u(\alpha_u(q)) = \dim_u \nu_q$ for every $q \in \mathbb{R}$.
5. *If $\mu \ne m_u$, then $\mathfrak{D}_u$ and T_u are real analytic strictly convex functions, and $(\mathfrak{D}_u, T_u)$ is a Legendre pair with respect to the variables α, q.*
6. *If $\mu = m_u$, then $d_{\mu,u}(x) = \dim_u \Sigma$ for every $x \in \Sigma$.*

Statements 1 through 5, with the expression "is a closed interval" replaced by "contains a closed interval" in Statement 3, are immediate consequences of results of Pesin and Weiss in [15]. In [16] Schmeling completed the proof of Statement 3.

For Statement 6 observe that $P_\Sigma(-\dim_u \Sigma \cdot u) = 0$ by Proposition 6.4, and since m_u is a Gibbs measure with potential $-\dim_u \Sigma \cdot u$ there are constants c_1, $c_2 > 0$ such that
$$c_1 \exp(-\dim_u \Sigma \cdot S_n u(x)) \le m_u(C_n(x)) \le c_2 \exp(-\dim_u \Sigma \cdot S_n u(x))$$
for every $n \in \mathbb{N}$ and $x \in \Sigma$. Since u is continuous and positive on the compact set Σ, if $\mu = m_u$ then
$$d_{\mu,u}(x) = \lim_{n\to\infty} -\frac{\log m_u(C_n(x))}{S_n u(x)} = \dim_u \Sigma$$

for every $x \in \Sigma$, and thus

$$K_\alpha = \begin{cases} \Sigma & \text{if } \alpha = \dim_u \Sigma \\ \emptyset & \text{if } \alpha \neq \dim_u \Sigma \end{cases}$$

and

$$\mathfrak{D}_u(\alpha) = \begin{cases} \dim_u \Sigma & \text{if } \alpha = \dim_u \Sigma, \\ 0 & \text{if } \alpha \neq \dim_u \Sigma. \end{cases}$$

In the particular case of the Hausdorff dimension this formula was obtained in [20].

Furthermore, one can prove that $K_{\alpha_u(q)}$ is σ-invariant and everywhere dense for every $q \in \mathbb{R}$. For one-sided subshifts the denseness follows immediately from the σ-invariance of $K_{\alpha_u(q)}$. For two-sided subshifts, note that $\sigma^{-1}|\Sigma$ is a subshift of finite type with transfer matrix equal to the transpose of that of $\sigma|\Sigma$, and thus $\sigma^{-1}|\Sigma$ is also topologically mixing. Since $K_{\alpha_u(q)} = \Sigma \cap \pi^{-1}(\pi K_{\alpha_u(q)})$, where $\pi\colon \{1,\dots,p\}^{\mathbb{Z}} \to \{1,\dots,p\}^{\mathbb{N}}$ is the canonical projection, we conclude that $K_{\alpha_u(q)}$ is everywhere dense.

We call m_u the **measure of maximal u-dimension** (in fact, m_u is the unique σ-invariant measure such that $\dim_u m_u = \dim_u \Sigma$), and ν_q the **full measure** for the spectrum $\mathfrak{D}_u$ at the point $\alpha_u(q)$, for each q.

Examples: 1. If $u \equiv 1$, the spectrum $\mathfrak{D}_u$ coincides with the entropy spectrum for local entropies introduced in [4], and m_u is the measure of maximal entropy.

2. If $u = \log a$ for some Hölder continuous function a, the spectrum $\mathfrak{D}_u$ coincides with the dimension spectrum for pointwise dimensions on a repeller of a $C^{1+\varepsilon}$ conformal expanding map f such that $a(x) = \|d_x f\|$ (expressed in terms of its underlying symbolic representation by a subshift of finite type), and m_u is the measure of maximal dimension. See [4] for details.

By Statement 2 in Theorem 6.6, one can set

$$\alpha_1 = \lim_{q\to+\infty} \alpha_u(q) \quad \text{and} \quad \alpha_2 = \lim_{q\to-\infty} \alpha_u(q).$$

For each interval $[a_1, a_2] \subset [\alpha_1, \alpha_2]$, set

$$K_{a_1,a_2} = \{x \in \Sigma\colon \underline{d}_{\mu,u}(x) = a_1 \text{ and } \overline{d}_{\mu,u}(x) = a_2\}.$$

The remaining statements in this section are immediate consequences of results of Schmeling in [16]. We always assume that $\sigma|\Sigma$ is a one-sided or two-sided topologically mixing subshift of finite type.

THEOREM 6.7: *The following properties hold:*

1. *We have* $\alpha_1 = \inf_{x\in\Sigma} \underline{d}_{\mu,u}(x)$ *and* $\alpha_2 = \sup_{x\in\Sigma} \overline{d}_{\mu,u}(x)$.
2. *For every real number* α, *we have* $K_\alpha = \emptyset$ *if and only if* $\alpha \notin [\alpha_1, \alpha_2]$.
3. *If* $\dim_u \Sigma \in [a_1, a_2] \subset [\alpha_1, \alpha_2]$, *then*

$$\dim_u K_{a_1,a_2} = \min\{\mathfrak{D}_u(a_1), \mathfrak{D}_u(a_2)\}.$$

Observe that Property 2 is an immediate consequence of Property 1. We note that $(\alpha_1, f(\alpha_1), \alpha_2, f(\alpha_2)) \in \mathbb{B}$, where $\mathbb{B}$ is the set

$$\{(x_1, y_1, x_2, y_2) \in \mathbb{R}^4\colon y_1 \le x_1 \le \dim_u \Sigma \text{ and } y_2 \le \dim_u \Sigma \le x_2\}.$$

THEOREM 6.8: *For each* (x_1, y_1, x_2, y_2) *in the interior of* $\mathbb{B}$ *there is a Hölder continuous function* ϕ *such that the spectrum* $\mathfrak{D}_u$ *with respect to the equilibrium measure of* ϕ *satisfies* $\alpha_i = x_i$ *and* $\mathfrak{D}_u(\alpha_i) = y_i$ *for* $i = 1$, 2.

Recall that $C_\theta(\Sigma)$ is the space of Hölder continuous functions on Σ with Hölder exponent θ, and for each $\phi \in C_\theta(\Sigma)$ one defines its norm by (3). The space $C_\theta(\Sigma)$ is a Baire space with the induced topology.

THEOREM 6.9: *There is a residual set* $\mathcal{R} \subset C_\theta(\Sigma)$ *such that* $\alpha_1 < \alpha_2$ *and* $\mathfrak{D}_u(\alpha_1) = \mathfrak{D}_u(\alpha_2) = 0$ *for the spectrum* $\mathfrak{D}_u$ *of every equilibrium measure of a potential in* $\mathcal{R}$.

7. Main results

7.1. IRREGULAR SETS AND SUBSHIFTS OF FINITE TYPE. Consider the sequences $F^i = \{f_n^i\colon \Sigma \to \mathbb{R}\}_{n\in\mathbb{N}}$ of strictly positive functions for $i = 1, \ldots, m$. We define the set $\mathfrak{F}(F^1, \ldots, F^m)$ by

$$\left\{x \in \Sigma\colon \lim_{n\to\infty} f_n^k(x) \text{ does not exist for } k = 1, \ldots, m\right\},$$

and call it the **irregular set** specified by the sequences of functions $F^1, \ldots, F^m$. Our concept of irregular set extends in a natural way the families of sets of "non-typical" points occurring naturally in the theory of dynamical systems. Namely, the sets $\mathfrak{B}(g)$, $\mathfrak{H}(\mu)$, $\mathfrak{D}(\mu)$, and $\mathfrak{L}_f$ are examples of irregular sets (see Sections 2, 3, and 4); for $\mathfrak{D}(\mu)$ this follows from (7). We will show, under mild assumptions, that any irregular set carries full topological entropy and full Hausdorff dimension.

The following is our main result for subshifts of finite type.

THEOREM 7.1: *Let $\sigma|\Sigma$ be a one-sided or two-sided topologically mixing subshift of finite type, $\phi_1, \ldots, \phi_m$ Hölder continuous functions on Σ, and g, u strictly positive Hölder continuous functions on Σ. The function ϕ_i is non-cohomologous to $\alpha_i g$ for every $i = 1, \ldots, m$, where α_i is the unique root of $P(\alpha_i g) = P(\phi_i)$, if and only if*

$$\dim_u \mathfrak{F}(\{S_n\phi_1/S_n g\}_{n\in\mathbb{N}}, \ldots, \{S_n\phi_m/S_n g\}_{n\in\mathbb{N}}) = \dim_u \Sigma.$$

The statement of Theorem 7.1 follows from the much more general statements formulated below. By using Markov partitions the proofs of the statements in Sections 3 and 4 can be reduced to Theorem 7.1; see Section 7.3 below.

7.2. IRREGULAR SETS AND DISTINGUISHING MEASURES. We now propose a general approach to estimate from below the u-dimension of irregular sets. This approach is based on the following concept. A collection of measures $\mu_1, \ldots, \mu_k$ is called **distinguishing** for $F^1, \ldots, F^m$ if for every $1 \le i \le m$, there exist distinct integers $j_1 = j_1(i)$, $j_2 = j_2(i) \in [1, k]$ and numbers $a^i_{j_1} \neq a^i_{j_2}$ such that

$$\lim_{n\to\infty} f^i_n(x) = a^i_{j_1} \quad \text{for } \mu_{j_1}\text{-almost all } x \in \Sigma,$$

$$\lim_{n\to\infty} f^i_n(x) = a^i_{j_2} \quad \text{for } \mu_{j_2}\text{-almost all } x \in \Sigma.$$

We can always assume that $k \le 2m$ in the definition. For example, let μ_1 and μ_2 be two distinct ergodic σ-invariant probability measures on Σ. Then, there is a function $g \in C(\Sigma)$ such that $\int_\Sigma g\, d\mu_1 \neq \int_\Sigma g\, d\mu_2$, and, by the Birkhoff Ergodic Theorem, the measures μ_1, μ_2 form a distinguishing collection for the sequence $\{S_n g/n\}_{n\in\mathbb{N}}$.

Let Z_Σ be the family of cylinders in Σ. We denote by CC' the cylinder corresponding to the juxtaposition of the tuples specifying C, $C' \in Z_\Sigma$, in this order. Recall that $C_n(x) \in Z_\Sigma$ denotes the cylinder of length n which contains the point $x \in \Sigma$. We denote by $|C|$ the length of the cylinder C.

With the help of distinguishing collections of measures we can obtain lower bounds for the u-dimension of irregular sets. We recall that a subshift $\sigma|\Sigma$ has the **specification property** if there exists a positive integer m such that for every C_1, $C_2 \in Z_\Sigma$ there exists $C \in Z_\Sigma$ of length m such that $C_1CC_2 \in Z_\Sigma$.

THEOREM 7.2: *If $\sigma|\Sigma$ is a one-sided or two-sided subshift with the specification property, $\mu_1, \ldots, \mu_k$ is a distinguishing collection of ergodic σ-invariant measures for $F^1, \ldots, F^m$, and u is a strictly positive Hölder continuous function on Σ, then*

$$\dim_u \mathfrak{F}(F^1, \ldots, F^m) \ge \min\{\dim_u \mu_1, \ldots, \dim_u \mu_k\}.$$

One can prove an analogous statement for arbitrary subshifts (see Theorem 7.2 below).

In order to effectively use the full power of Theorem 7.2, one needs to know when there exist distinguishing collections of measures. The following statement solves this problem for subshifts of finite type. Recall the notion of full measure introduced in Section 6.2.

THEOREM 7.3: *Let $\sigma|\Sigma$ be a one-sided or two-sided topologically mixing subshift of finite type, $\phi_1, \dots, \phi_m$ Hölder continuous functions on Σ, and g, u strictly positive Hölder continuous functions on Σ. If for each $i = 1, \dots, m$ the function ϕ_i is non-cohomologous to $\alpha_i g$, where α_i is the unique root of $P(\alpha_i g) = P(\phi_i)$, then, for every $\varepsilon > 0$, there exist ergodic σ-invariant measures $\mu_1, \dots, \mu_m$ such that:*

1. *$\mu_1, \dots, \mu_m$ are full measures for the spectrum $\mathfrak{D}_u$;*
2. *$\mu_1, \dots, \mu_m$, m_u is a distinguishing collection of measures for the sequences of functions $\{S_n\phi_1/S_ng\}_{n\in\mathbb{N}}, \dots, \{S_n\phi_m/S_ng\}_{n\in\mathbb{N}}$;*
3. *$\min\{\dim_u \mu_1, \dots, \dim_u \mu_m\} > \dim_u \Sigma - \varepsilon$.*

7.3. IRREGULAR SETS AND MARKOV PARTITIONS. Let M be a smooth manifold, and $f\colon M \to M$ a topologically mixing C^1 map. We consider a subset $X \subset M$ and assume either that $f|X$ is a conformal expanding map or that $f|X$ is a conformal hyperbolic diffeomorphism (see Sections 3 and 4). We fix a Markov partition and its corresponding coding map $\chi\colon \Sigma \to X$.

The following notion is crucial in our approach. A measure μ on X is called **diametrically regular** if there exist constants $\tau > 1$ and $c > 0$ such that $\mu(B(y, \tau r)) \le c\mu(B(y, r))$ for any point $y \in X$ and any $r > 0$.

Examples: 1. If $f|X$ is a topologically mixing subshift of finite type, then any Gibbs measure μ having a Hölder continuous potential is diametrically regular; furthermore one can easily show that for each $\tau > 1$ there exists $c > 0$ such that $\mu(B(x, \tau r)) \le c\mu(B(x, r))$ for any point $x \in X$ and any $r > 0$.

2. In [13], Pesin showed that for repellers of conformal maps any equilibrium measure having a Hölder continuous potential is diametrically regular. Similarly, for locally maximal hyperbolic sets of conformal diffeomorphisms he showed that any equilibrium measure having a Hölder continuous potential, as well as their conditionals on stable and unstable manifolds, are diametrically regular.

Recall that ν_q denotes the full measure for the spectrum $\mathfrak{D}_{u\circ\chi}$ (note that the function $u \circ \chi$ is Hölder continuous) supported on $K_{\alpha_{u\circ\chi}(q)}$ (see Section 6.2). We define an f-invariant measure on X by $\lambda_q = \nu_q \circ \chi^{-1}$.

THEOREM 7.4: *If u is a strictly positive Hölder continuous function on X, and μ is a diametrically regular equilibrium measure on X having a Hölder continuous potential, then:*

1. *for every $q \in \mathbb{R}$, $\lambda_q(\chi(K_{\alpha_{u\circ\chi}(q)})) = 1$, and*

$$\dim_u \chi(K_{\alpha_{u\circ\chi}(q)}) = \dim_u \lambda_q = \dim_{u\circ\chi} \nu_q = \mathfrak{D}_{u\circ\chi}(\alpha_{u\circ\chi}(q));$$

2. *for every $q \in \mathbb{R}$, $d_{\lambda_q,u}(y) = \dim_u \lambda_q$ for λ_q-almost all $y \in \chi(K_{\alpha_{u\circ\chi}(q)})$, and $\overline{d}_{\lambda_q,u}(y) \le \dim_u \lambda_q$ for every $y \in \chi(K_{\alpha_{u\circ\chi}(q)})$;*
3. *for every $\alpha \in [\alpha_1, \alpha_2]$ the set $\chi(K_\alpha)$ coincides with the set of points $y \in X$ such that*

$$\lim_{\operatorname{diam}\mathfrak{U}\to 0} \underline{d}_{\mu,u}(y,\mathfrak{U}) = \lim_{\operatorname{diam}\mathfrak{U}\to 0} \overline{d}_{\mu,u}(y,\mathfrak{U}) = \alpha;$$

4. *the set $\chi(\{x \in \Sigma\colon \underline{d}_{\mu,u\circ\chi}(x) < \overline{d}_{\mu,u\circ\chi}(x)\}) = X \setminus \bigcup_{\alpha\in[\alpha_1,\alpha_2]} \chi(K_\alpha)$ coincides with the set of points $y \in X$ such that*

$$\lim_{\operatorname{diam}\mathfrak{U}\to 0} \underline{d}_{\mu,u}(y,\mathfrak{U}) < \lim_{\operatorname{diam}\mathfrak{U}\to 0} \overline{d}_{\mu,u}(y,\mathfrak{U}).$$

This shows that for diametrically regular measures the multifractal properties of $f|X$ are inherited from those of the associated symbolic dynamics $\sigma|\Sigma$.

THEOREM 7.5: *Assume that $f|X$ is a conformal expanding map. Let $\phi_1, \dots, \phi_m$ be Hölder continuous functions on X, and g a strictly positive Hölder continuous function on X. The function ϕ_i is non-cohomologous to $\alpha_i g$ for every $i = 1, \dots, m$, where α_i is the unique root of $P(\alpha_i g) = P(\phi_i)$, if and only if*

$$h(f|\mathfrak{F}(\{S_n\phi_1/S_n g\}_{n\in\mathbb{N}}, \dots, \{S_n\phi_m/S_n g\}_{n\in\mathbb{N}}) = h(f|X)$$

and

$$\dim_H \mathfrak{F}(\{S_n\phi_1/S_n g\}_{n\in\mathbb{N}}, \dots, \{S_n\phi_m/S_n g\}_{n\in\mathbb{N}}) = \dim_H X.$$

This result indicates that the boundaries of Markov partitions have no influence in the study of the entropy and Hausdorff dimension of irregular sets of repellers. The case of hyperbolic sets is considered in the proof of Theorem 4.2.

7.4. IRREGULAR SETS AND ARBITRARY SUBSHIFTS. In order to extend the above results to arbitrary subshifts we need to introduce additional assumptions.

Consider a non-decreasing sequence $\Psi = \{\psi_n\}_{n\in\mathbb{N}}$ of positive integers such that

$$\psi_n/n \to 0 \quad \text{as } n \to \infty. \tag{14}$$

Define the subset $\Sigma_\Psi \subset \Sigma$ of points $x \in \Sigma$ such that for each $n \in \mathbb{N}$ and $C \subset Z_\Sigma$ with $|C| < \psi_n$ and $CC_n(x) \in Z_\Sigma$, if $\overline{C} \in Z_\Sigma$ then there exists $\underline{C} \in Z_\Sigma$ such that

$$(15) \qquad CC_n(x)\underline{C}\overline{C} \in Z_\Sigma \quad \text{and} \quad |\underline{C}| \leq |CC_n(x)| + \psi_{|CC_n(x)|}.$$

We note that $\Sigma_\Psi \subset \sigma\Sigma_\Psi$, but presumably Σ_Ψ need not be σ-invariant in general. The conditions in (15) indicate that one can construct a cylinder with any prescribed initial and final symbols; moreover, this can be done in such a way that the connecting symbols between the initial symbols C and final symbols $\overline{C}$ is approximately of order $|C|$, i.e., the order of the length of the initial symbols.

For each measure μ on Σ we consider the following property:

$$(16) \qquad \text{There exists a sequence } \Psi \text{ such that } \mu(\Sigma_\Psi) > 0.$$

This holds, for example, for σ-invariant measures on subshifts of finite type, sofic subshifts, and, more generally, subshifts with the specification property; in each of these cases $\Sigma_\Psi = \Sigma$ for some constant sequence Ψ.

THEOREM 7.6: *Let $\sigma|\Sigma$ be a one-sided or two-sided subshift, and u a strictly positive Hölder continuous function on Σ. If $\mu_1, \ldots, \mu_k$ is a distinguishing collection of ergodic σ-invariant measures for F^1, …, F^m such that the condition (16) holds for each measure μ_i with respect to some sequence Ψ^i satisfying (14), then*

$$\dim_u \mathfrak{F}(F^1, \ldots, F^m) \geq \min\{\dim_u \mu_1, \ldots, \dim_u \mu_k\}.$$

Furthermore, given $\varepsilon > 0$ there exist a set $\Lambda \subset \mathfrak{F}(F^1, \ldots, F^m)$, and a measure μ on Σ with $\mu(\Lambda) > 0$, such that if $x \in \Lambda$ then

$$\underline{d}_{\mu,u}(x) \geq \min\{\dim_u \mu_1, \ldots, \dim_u \mu_k\} - \varepsilon.$$

It is an open question to describe the class of subshifts which possess distinguishing collections of measures.

The following is a subproduct of the proof of Theorem 7.2.

THEOREM 7.7: *Let $\mu_1, \ldots, \mu_k$ be ergodic σ-invariant measures such that the condition (16) holds for each measure μ_i with respect to some sequence Ψ^i satisfying (14). If not all the numbers $\dim_u \mu_1$, …, $\dim_u \mu_k$ are equal, then, for any strictly positive Hölder continuous function u on Σ, we have*

$$(17) \quad \dim_u \bigcap \mathfrak{F}(\{-\log \mu_i(C_n(\cdot))/S_n u\}_{n\in\mathbb{N}}) \geq \min\{\dim_u \mu_1, \ldots, \dim_u \mu_k\},$$

where the intersection is taken over all i such that

$$\dim_u \mu_i < \max\{\dim_u \mu_1, \ldots, \dim_u \mu_k\}.$$

An immediate consequence is the following.

THEOREM 7.8: *Let $\sigma|\Sigma$ be a one-sided or two-sided subshift with the specification property, $\mu_1, \dots, \mu_k$ ergodic σ-invariant measures, and u a strictly positive Hölder continuous function on Σ. If not all the numbers $\dim_u \mu_1, \dots, \dim_u \mu_k$ are equal, then the inequality (17) holds.*

8. Proofs

8.1. PROOFS OF THE RESULTS IN SECTION 7. We first formulate some auxiliary results.

PROPOSITION 8.1: *If μ_1 and μ_2 are probability measures on Σ, and u is a strictly positive Hölder continuous function on Σ, then, for every $\delta > 0$,*

$$\mu_1(\{x \in \Sigma\colon \underline{d}_{\mu_2,u}(x) > \dim_u \mu_1 - \delta\}) > 0. \tag{18}$$

Proof of Proposition 8.1: If (18) does not hold, then the set

$$\Gamma_\delta = \{x \in \Sigma\colon \underline{d}_{\mu_2,u}(x) \le \dim_u \mu_1 - \delta\} \tag{19}$$

has full μ_1-measure. For each $x \in \Gamma_\delta$, let $\{n_k(x)\}_{k\in\mathbb{N}}$ be an increasing sequence of positive integers such that

$$-\frac{\log \mu_2(C_{n_k(x)}(x))}{S_{n_k(x)}u(x)} \le \dim_u \mu_1 - \delta/2$$

for each k. Observe that two cylinders are either disjoint, or one is contained in the other. Hence, for each $L > 0$ there is a finite or countable cover $\{C_{m_i}(x_i)\colon i \in \mathbb{N}\}$ of Γ_δ formed by disjoint cylinders, for some points $x_i \in \Gamma_\delta$ and integers $m_i \in \{n_k(x_i)\colon k \in \mathbb{N}\}$ such that $m_i > L$ for each $i \in \mathbb{N}$. We obtain

$$\begin{aligned}\mu_2(\Gamma_\delta) &= \sum_{i=1}^{\infty} \mu_2(C_{m_i}(x_i))\\ &\ge \sum_{i=1}^{\infty} \exp[-(\dim_u \mu_1 - \delta/2)S_{m_i}u(x_i)]\\ &\ge c\sum_{i=1}^{\infty} \sup_{x\in C_{m_i}(x_i)} \exp[-(\dim_u \mu_1 - \delta/2)S_{m_i}u(x)],\end{aligned}$$

where c is a constant depending only on the Hölder exponent of u. Hence, $\dim_u \mu_1 - \delta/2 \ge \dim_u \Gamma_\delta \ge \dim_u \mu_1$, because $\mu_1(\Gamma_\delta) = 1$. This contradiction implies the desired result. ∎

COROLLARY 8.2: *Let μ_1 and μ_2 be two probability measures on Σ, and u a strictly positive Hölder continuous function on Σ. If μ_1 is an ergodic σ-invariant measure, then*

$$\mu_1(\{x \in \Sigma\colon \underline{d}_{\mu_2,u}(x) \geq \dim_u \mu_1\}) = 1.$$

Proof of Corollary 8.2: For each $\delta > 0$, the set Γ_δ defined by (19) is σ-invariant. By Proposition 8.1, $\mu_1(\Sigma \setminus \Gamma_\delta) = 1$ for every $\delta > 0$, and hence the set

$$\bigcap_{\delta>0}(\Sigma \setminus \Gamma_\delta) = \{x \in \Sigma\colon \underline{d}_{\mu_2,u}(x) \geq \dim_u \mu_1\}$$

has also full μ_1-measure. ∎

Proof of Theorem 7.3: For each i, we have

$$\lim_{n\to\infty} -\frac{S_n\phi_i(x)}{S_n g(x)} = -\frac{\int_\Sigma \phi_i\, dm_u}{\int_\Sigma g\, dm_u} \quad \text{for } m_u\text{-almost every } x \in \Sigma.$$

Fix $\varepsilon > 0$. Since ϕ_i is not cohomologous to $\alpha_i g$, one can show that for each $\alpha > 0$ the set of points $q \in [-\alpha, \alpha]$ such that $\int_\Sigma \varphi_i\, d\nu_q = \alpha_i \int_\Sigma g\, d\nu_q$ is finite. Otherwise, by the analytic dependence of $\int_\Sigma \varphi_i\, d\nu_q$ and $\int_\Sigma g\, d\nu_q$ on q, we would have $\int_\Sigma \varphi_i\, d\mu = \alpha_i \int_\Sigma g\, d\mu$ for every equilibrium measure μ with potential in a $C_\theta(\Sigma)$ open neighborhood of some φ_q, and hence $\int_\Sigma \varphi_i\, d\mu = \alpha_i \int_\Sigma g\, d\mu$ for every Gibbs measure. But this is impossible because φ_i is non-cohomologous to $\alpha_i g$. Thus, by Theorem 6.6, there is an equilibrium measure ν_i such that $\dim_u \nu_i > \dim_u \Sigma - \varepsilon$, and

$$\lim_{n\to\infty} -\frac{S_n\phi_i(x)}{S_n g(x)} \neq -\frac{\int_\Sigma \phi_i\, dm_u}{\int_\Sigma g\, dm_u} \quad \text{for } \nu_i\text{-almost every } x \in \Sigma.$$

The collection of $m+1$ measures $\nu_1, \ldots, \nu_m$, and m_u gives rise to a collection of distinguishing measures with the desired properties. ∎

Proof of Theorem 7.6: For the sake of clarity we first present the proof in the case $m = 1$. The general case will be discussed at the end.

When $m = 1$, we write $f_n = f_n^1$ for each $n \in \mathbb{N}$, and, without loss of generality, we may assume that μ_1, μ_2 is a distinguishing collection of measures for $F = \{f_n\}_{n\in\mathbb{N}}$ with $\dim_u \mu_1 \geq \dim_u \mu_2$; we write $a_j^1 = a_j$ for $j = 1, 2$. We may also assume that $a_j \neq 0$ for $j = 1, 2$. Otherwise we can consider the sequence of functions $F + a = \{f_n + a\}_{n\in\mathbb{N}}$, where a is a non-zero constant, since $\mathfrak{F}(F + a) = \mathfrak{F}(F)$. Choose a positive number δ such that

$$|a_1 - a_2| > 4\delta. \tag{20}$$

We consider the sequence $\Psi = \{\max\{\psi_n^1, \psi_n^2\}\}_{n\in\mathbb{N}}$, where $\Psi^i = \{\psi_n^i\}_{n\in\mathbb{N}}$ for $i = 1, 2$. For each integer $s \geq 1$, we set

$$p_s = \begin{cases} 1 & \text{if } s \text{ is odd,} \\ 2 & \text{if } s \text{ is even.} \end{cases}$$

For each integer $\ell \geq 1$, let $\widehat{\Gamma}_1^\ell \subset \Sigma_\Psi$ be the set of points $x \in \Sigma_\Psi$ such that for all $n \geq \ell$ and $i = 1, 2$, we have

$$|f_n(x) - a_1| < \delta \quad \text{and} \quad -\frac{\log \mu_i(C_n(x))}{S_n u(x)} > \dim_u \mu_1 - \delta. \tag{21}$$

For each $\ell \geq 1$, let $\widehat{\Gamma}_2^\ell \subset \Sigma_\Psi$ be the set of points $x \in \Sigma_\Psi$ such that for all $n \geq \ell$,

$$|f_n(x) - a_2| < \delta \quad \text{and} \quad -\frac{\log \mu_2(C_n(x))}{S_n u(x)} > \dim_u \mu_2 - \delta. \tag{22}$$

Clearly $\widehat{\Gamma}_i^{\ell+1} \supset \widehat{\Gamma}_i^\ell$ for each $\ell \geq 1$, and $i = 1, 2$.

Let ν_1 and ν_2 be the normalized measures obtained from the restrictions of μ_1 and μ_2 to the set Σ_Ψ. Fix $\varepsilon \in (0,1)$, and for each integer $s \geq 1$ set

$$\ell_s = \min\left(\left\{\ell \in \mathbb{N}\colon \nu_{p_s}(\widehat{\Gamma}_{p_s}^\ell) > 1 - \varepsilon/2^{s+1}\right\} \cup \{\ell_{s-1}\}\right),$$

where $\ell_0 = \infty$. We note that $\ell_s \geq \ell_{s-1}$. By Corollary 8.2 and Theorem 6.3, we have $\ell_s < \infty$ for every $s \geq 1$.

For $j = 1, 2$, since μ_j is σ-invariant, the set of points $x \in \Sigma$ such that

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} f_n(\sigma^m x)$$

for every $m \in \mathbb{N}$ has full μ_j-measure. We define the number

$$D_{n,m}(x) = \max\{|f_{n+m}(y)/f_m(x)|, |f_m(x)/f_{n+m}(z)|\colon y, z \in \sigma^{-n}x\}.$$

By Lusin's Theorem, for each $j = 1, 2$, and $\delta > 0$ there is an integer $r_j(n, \delta) \geq n$ such that $D_{n,m}(x) < 1 + \delta$ for all $m > r_j(n, \delta)$ and all x outside a set $Y_j^n(\varepsilon)$ of μ_j-measure at least $1 - \delta$.

For each $s \geq 1$, we define inductively the increasing sequences of positive integers $\{n_s\}_{s\in\mathbb{N}}$ and $\{m_s\}_{s\in\mathbb{N}}$ by $m_1 = n_1 = \ell_1$, and, for every $s \geq 2$, by

$$\begin{aligned} m_s &= r_{p_s}(n_{s-1} + \psi_{n_{s-1}}, \varepsilon/2^{s+1}) + \ell_{s+1}! \\ n_s &= n_{s-1} + \psi_{n_{s-1}} + m_s + 1. \end{aligned} \tag{23}$$

We set

$$\Gamma_{p_s}^{\ell_s} = \widehat{\Gamma}_{p_s}^{\ell_s} \cap Y_{p_s}^{n_{s-1}}(\varepsilon/2^{s+1}).$$

Then

$$\nu_{p_s}(\Gamma_{p_s}^{\ell_s}) > 1 - \varepsilon/2^s. \tag{24}$$

For each $s \geq 1$, we define a family of cylinders by

$$\mathfrak{C}_s = \{C_{m_s}(x) \colon x \in \Gamma_{p_s}^{\ell_s}\}; \tag{25}$$

moreover, we set $\mathfrak{D}_1 = \mathfrak{C}_1$, and

$$\mathfrak{D}_s = \{\underline{C}C\overline{C} \in Z_\Sigma \colon \underline{C} \in \mathfrak{D}_{s-1},\ \overline{C} \in \mathfrak{C}_s, \text{ and } C \in Z_\Sigma \text{ is minimal}\}. \tag{26}$$

Here, minimality refers to the order $<$ in Z_Σ defined by: if C, $C' \in Z_\Sigma$ are distinct, we write $C < C'$ if $|C| < |C'|$, or if $|C| = |C'|$ but C is smaller than C' in the lexicographical order. We note that if $\sigma|\Sigma$ has the specification property, then the length of C in (26) may be taken constant.

We now prove that for each $\underline{C}C\overline{C} \in \mathfrak{D}_s$ with $\underline{C} \in \mathfrak{D}_{s-1}$ and $\overline{C} \in \mathfrak{C}_s$, we have $|\underline{C}| \leq n_{s-1}$ and $|C| < \psi_{n_{s-1}}$ for each $s \geq 2$. For $s = 2$ this is clear because $n_1 = m_1$. Using (23) and induction on $s > 2$, we obtain

$$|\underline{C}C\overline{C}| \leq n_{s-1} + \psi_{n_{s-1}} + m_s < n_s,$$

and hence $|C'| \leq \psi_{n_s}$ for each $\underline{C}'C'\overline{C}' \in \mathfrak{D}_{s+1}$ with $\underline{C}' \in \mathfrak{D}_s$ and $\overline{C}' \in \mathfrak{C}_{s+1}$, because Ψ is non-decreasing.

Set

$$\Lambda = \bigcap_{s \geq 1} \bigcup_{C \in \mathfrak{D}_s} C. \tag{27}$$

We define a measure μ on Λ by $\mu(C) = \nu_1(C)$ if $C \in \mathfrak{D}_1$, by

$$\mu(\underline{C}C\overline{C}) = \mu(\underline{C})\nu_{p_s}(\overline{C}) \tag{28}$$

if $\underline{C}C\overline{C} \in \mathfrak{D}_s$ for some $s > 1$, and arbitrarily for backward cylinders, i.e., cylinders with coordinates fixed in the past. We extend μ to Σ by $\mu(A) = \mu(A \cap \Lambda)$ for each measurable set $A \subset \Sigma$. For each $s \geq 1$ and every $\underline{C} \in \mathfrak{D}_{s-1}$, it follows from (24) that

$$\mu\left(\bigcup_{\overline{C} \in \mathfrak{D}_s} \underline{C} \cap \overline{C}\right) \geq \mu(\underline{C})\left(1 - \frac{\varepsilon}{2^s}\right),$$

and hence

$$\mu(\Lambda) \geq \prod_{s=1}^{\infty}\left(1 - \frac{\varepsilon}{2^s}\right) > 0$$

for all sufficiently small ε.

Let now $x \in C \in \mathfrak{D}_s$. Then $m_s \le |C| \le n_s$, and $\sigma^{|C|-m_s}x \in \Gamma_{p_s}^{\ell_s}$ for each $s \ge 1$. By (21) and (22), we obtain

$$\begin{aligned}|f_{|C|}(x) - a_{p_s}| \le & |f_{m_s}(\sigma^{|C|-m_s}x) - a_{p_s}| \times f_{|C|}(x)/f_{m_s}(\sigma^{|C|-m_s}x) \\ & + |1 - f_{|C|}(x)/f_{m_s}(\sigma^{|C|-m_s}x)| \times |a_{p_s}| \\ \le & D_{|C|-m_s,m_s}(x) \times |f_{m_s}(\sigma^{|C|-m_s}x) - a_{p_s}| \\ & + (D_{|C|-m_s,m_s}(x) - 1) \times |a_{p_s}|.\end{aligned}$$

Hence, for all sufficiently large s and every $x \in C \in \mathfrak{D}_s$, we have

$$|f_{|C|}(x) - a_{p_s}| < 2\delta. \tag{29}$$

It follows from (20) and (29) that

$$\mathfrak{F}(F) \supset \Lambda. \tag{30}$$

Lemma 8.3: *If $x \in \Lambda$, then*

$$\liminf_{n\to\infty} -\frac{\log \mu(C_n(x))}{S_n u(x)} \ge \dim_u \mu_2 - 3\delta.$$

Proof of the lemma: Let $x \in \Lambda$. For each $q \in \mathbb{N}$, choose an integer s_q such that $|C^{s_q}| \le q < |C^{s_q+1}|$, where

$$\mathfrak{D}_{s_q+1} \ni C^{s_q+1} \subset C_q(x) \subset C^{s_q} \in \mathfrak{D}_{s_q}.$$

Assume that

$$|C^{s_q}| \le q \le |C^{s_q}| + \psi_{|C^{s_q}|} + \ell_{s_q+1}. \tag{31}$$

We have $(\psi_{|C^{s_q}|} + \ell_{s_q+1})/|C^{s_q}| \to 0$ as $q \to \infty$, and hence

$$\begin{aligned}\frac{S_q u(x)}{S_{|C^{s_q}|}u(x)} &\le \frac{S_{|C^{s_q}|+\psi_{|C^{s_q}|}+\ell_{s_q+1}}u(x)}{S_{|C^{s_q}|}u(x)} \\ &\le 1 + \frac{\psi_{|C^{s_q}|} + \ell_{s_q+1}}{|C^{s_q}|} \times \frac{\max_{x\in\Sigma} u(x)}{\min_{x\in\Sigma} u(x)} \to 1,\end{aligned}$$

as $q \to \infty$. Therefore there exists $q_1 \in \mathbb{N}$ such that

$$\begin{aligned}-\frac{\log \mu(C_q(x))}{S_q u(x)} &\ge -\frac{\log \mu(C^{s_q})}{S_q u(x)} \\ &\ge -\frac{\log \mu(C^{s_q})}{S_{|C^{s_q}|}u(x)} \times \frac{S_{|C^{s_q}|}u(x)}{S_q u(x)} \\ &\ge \dim_u \mu_2 - 2\delta\end{aligned} \tag{32}$$

for every $q \geq q_1$. In particular,

$$-\frac{\log \mu(C^{s_q})}{S_{|C^{s_q}|}u(x)} \geq \dim_u \mu_2 - 2\delta \tag{33}$$

for every $q \geq q_1$. When (31) does not hold, we have

$$\mu(C_q(x)) = \mu(C^{s_q})\nu_{p_{s_q}}(\widetilde{C}) \leq \mu(C^{s_q})\mu_{p_{s_q+1}}(\widetilde{C}),$$

where $C_q(x) = C^{s_q}C\widetilde{C}$ and the cylinder $\widetilde{C}$ contains an element of $\mathfrak{C}_{s_q+1}$; moreover, $|C| < \psi_{|C^{s_q}|}$ and $|\widetilde{C}| > \ell_{s_q+1}$. Thus

$$|C^{s_q}| + |\widetilde{C}| \leq q \leq |C^{s_q}| + \psi_{|C^{s_q}|} + |\widetilde{C}|,$$

and

$$\frac{S_{|C^{s_q}|}u(x) + S_{|\widetilde{C}|}u(\sigma^{q-|\widetilde{C}|}x)}{S_q u(x)} \to 1$$

as $q \to \infty$. Therefore, by the definition of $\Gamma^{\ell_{s_q+1}}_{p_{s_q+1}}$, and (33), there exists $q_2 \geq q_1$ such that

$$\begin{aligned}
-\frac{\log \mu(C_q(x))}{S_q u(x)} &\geq \frac{1}{S_q u(x)}\left(-\log \mu(C^{s_q}) - \log \mu_{p_{s_q+1}}(\widetilde{C})\right)\\
&\geq \frac{S_{|C^{s_q}|}u(x)(\dim_u \mu_2 - 2\delta) + S_{|\widetilde{C}|}u(\sigma^{q-|\widetilde{C}|}x)(\dim_u \mu_2 - \delta)}{S_q u(x)}\\
&\geq \frac{S_{|C^{s_q}|}u(x) + S_{|\widetilde{C}|}u(\sigma^{q-|\widetilde{C}|}x)}{S_q u(x)}(\dim_u \mu_2 - 2\delta)\\
&\geq \dim_u \mu_2 - 3\delta
\end{aligned} \tag{34}$$

for every $q \geq q_2$. The desired statement follows now immediately from (32) and (34). ∎

By Theorem 3.1 in [13], and Lemma 8.3, we obtain

$$\dim_u \Lambda \geq \dim_u(\mu|\Lambda) \geq \dim_u \mu_2 - 3\delta. \tag{35}$$

By (30) and since δ is arbitrary,

$$\dim_u \mathfrak{F}(F) \geq \dim_u \mu_2.$$

Since $\dim_u \mu_1 \geq \dim_u \mu_2$, this completes the proof of the theorem in the case $m = 1$.

We now briefly discuss how to deal with the case $m > 1$. We consider the sequence $\Psi = \{\max\{\psi_n^1, \dots, \psi_n^k\}\}_{n\in\mathbb{N}}$, where $\Psi^i = \{\psi_n^i\}_{n\in\mathbb{N}}$ for $i = 1, \dots, k$. For each integer $s \geq 1$, we set $p_s = s \pmod{k} + 1$.

Without loss of generality, we may assume that

$$\dim_u \mu_{j_1(i)} \geq \dim_u \mu_{j_2(i)} \quad \text{for all } 1 \leq i \leq m,$$

and

$$\dim_u \mu_j \geq \dim_u \mu_k \quad \text{for all } 1 \leq j \leq k.$$

For each integer $\ell \geq 1$, and $i = 1, \dots, m$, let $\widehat{\Gamma}^\ell_{i,j_1(i)} \subset \Sigma_\Psi$ be the set of points $x \in \Sigma_\Psi$ such that for all $n \geq \ell$ and $t = k$, $j_1(i)$, we have

$$|f_n^i(x) - a^i_{j_1(i)}| < \delta \quad \text{and} \quad -\frac{\log \mu_t(C_n(x))}{S_n u(x)} > \dim_u \mu_{j_1(i)} - \delta.$$

For each $\ell \geq 1$, let $\widehat{\Gamma}^\ell_{i,j_2(i)} \subset \Sigma_\Psi$ be the set of points $x \in \Sigma_\Psi$ such that for all $n \geq \ell$ and $t = k$, $j_2(i)$,

$$|f_n^i(x) - a^i_{j_2(i)}| < \delta \quad \text{and} \quad -\frac{\log \mu_t(C_n(x))}{S_n u(x)} > \dim_u \mu_{j_2(i)} - \delta.$$

We then define a set $\Lambda \subset \Sigma$ in a similar way to that for $m = 1$, selecting alternatively cylinders from $\Gamma_{1,j_1(1)}, \Gamma_{1,j_2(1)}, \Gamma_{2,j_1(2)}, \Gamma_{2,j_2(2)}, \dots, \Gamma_{m,j_1(m)}$, and $\Gamma_{m,j_2(m)}$ (not necessarily in this order; compare with (25) and (26)). The remaining arguments are similar. ∎

We remark that in the case of two-sided subshifts the cylinders used to construct the set Λ in (27) are forward cylinders, i.e., they are completely determined by a finite number of symbols in the future. Moreover the non-invariant measure μ constructed in (28) can be, for the purposes of the proof of Theorem 7.6, arbitrarily defined for backward cylinders, primarily since we only require Lemma 8.3 when $n \to +\infty$.

One can also consider "two-sided" irregular sets of points, that is, irregular sets for which there exist no limits both when $n \to \infty$ and when $n \to -\infty$. In this new situation one can obtain a similar statement to that in Theorem 7.6, with some slight modifications in the proof. Namely, the set Λ defined by (27) must be replaced by $\Lambda \cap \Lambda^-$, where Λ^- is constructed in a similar way to that of Λ replacing forward cylinders in (25) by backward cylinders, and inverting the order of the cylinders in (26). Furthermore, the measure μ is defined in (28) for forward cylinders, but instead of choosing it arbitrarily for backward cylinders in $\Lambda \cap \Lambda^-$ we must also define the μ-measure of a backward cylinder by taking an alternated product of the measures ν_1 and ν_2 (note that these are invariant measures, and hence backward and forward typical points coincide in sets of full measure) as in (28). Using these modified versions one can prove a stronger version of Lemma 8.3: if $x \in \Lambda \cap \Lambda^-$, then

$$\liminf_{n\to\pm\infty} -\frac{\log \mu(C_n(x))}{\sum_{k=0}^n u(\sigma^k x)} \geq \dim_u \mu_2 - 3\delta.$$

Using this property one can show (for two-sided shifts) that the set of points where both the backward and forward averages do not converge has also full u-dimension.

Proof of Theorems 7.1 and 7.2: These are immediate consequences of Theorems 7.3 and 7.6. ∎

Proof of Theorem 7.4: One can easily obtain Statements 1, 2, and 3 by repeating with slight changes the proof of Theorem 21.1 in [13] (and in particular that of Lemmas 2 and 3 in Theorem 21.1). More precisely, Statements 1 and 2 are obtained as in Lemma 2, while Statement 3 is obtained as in Lemma 3.

Statement 4 is not a consequence of Statement 3 because we did not discard the hypothesis that there exists $x \in \Sigma$ such that $\underline{d}_{\mu,u\circ\chi}(x) < \overline{d}_{\mu,u\circ\chi}(x)$ and

$$\lim_{\operatorname{diam}\mathfrak{U}\to 0} \underline{d}_{\mu,u}(\chi(x),\mathfrak{U}) = \lim_{\operatorname{diam}\mathfrak{U}\to 0} \overline{d}_{\mu,u}(\chi(x),\mathfrak{U}). \tag{36}$$

However, it is an immediate consequence of the arguments in the proof of the above mentioned Lemma 3 in [13] that $\underline{d}_{\mu,u\circ\chi}(x) = \overline{d}_{\mu,u\circ\chi}(x)$ (i.e., there exists the limit corresponding to the pointwise dimension of x in Σ) if and only if the identity (36) holds (i.e., if and only if there exists the limit corresponding to the pointwise dimension of $\chi(x)$ in X). That is, either both limits exist or both limits do not exist. This completes the proof of the theorem. ∎

Proof of Theorem 7.5: It follows from Statements 3 and 4 in Theorem 7.4 that the irregular set

$$\mathfrak{F} = \mathfrak{F}(\{S_n\phi_1/S_ng\}_{n\in\mathbb{N}}, \ldots, \{S_n\phi_m/S_ng\}_{n\in\mathbb{N}})$$

coincides with the image by χ of the corresponding irregular set $\mathfrak{F}' \subset \Sigma$ for the associated symbolic dynamics $\sigma|\Sigma$; furthermore $\chi^{-1}\mathfrak{F} = \mathfrak{F}'$. For the topological entropy the identity can be obtained as in the proof of Theorem 3.2 (see below), and so we will not reproduce the proof here.

We now consider Hausdorff dimension. The Hausdorff dimension of sets in X coincides with the $\log a$-dimension, where $a(x) = \|d_x f\|$ (see Section 6.1).

Proceeding as in [13, p. 200], for each $r > 0$ sufficiently small we obtain a Moran cover $\mathfrak{U}_r$ of X. This is a special cover composed of images under χ of cylinders, such that given $x \in X$ and a sufficiently small $r > 0$ the number of sets in the cover that have non-empty intersection with $B(x, r)$ is bounded from above by a number κ, which is independent of x and r.

We now equip Σ with the unique metric d_a such that a cylinder C of length n has diameter $\sup_{x\in C} \prod_{k=0}^{n-1}(a(\chi(\sigma^k x))^{-1})$. The Hausdorff dimension associated to this metric coincides with the $(\log a \circ \chi)$-dimension in Σ. Let $\mathfrak{U}$ be a finite cover of $\mathfrak{F}$ by open balls. Then for each $B \in \mathfrak{U}$ of radius r there are at most κ cylinders (not necessarily all with the same length) such that their image by χ are the elements of the Moran cover $\mathfrak{U}_r$ intersecting B. Doing the same for every $B \in \mathfrak{U}$, we obtain a family $\mathfrak{V}$ of cylinders in Σ which form a cover of $\mathfrak{F}'$, and

$$\sum_{U\in\mathfrak{U}} (\operatorname{diam} U)^s \leq \sum_{C\in\mathfrak{V}} (\operatorname{diam} C)^s \leq \kappa \sup a^s \sum_{U\in\mathfrak{U}} (\operatorname{diam} U)^s.$$

Therefore $\dim_{\log a\circ\chi} \mathfrak{F}' = \dim_H \mathfrak{F}$. By repeating this argument with $\mathfrak{F}$ replaced by X we conclude that $\dim_{\log a\circ\chi} \Sigma = \dim_H X$. By Theorem 7.1 we have $\dim_{\log a\circ\chi} \mathfrak{F}' = \dim_{\log a\circ\chi} \Sigma$, and hence $\dim_H \mathfrak{F} = \dim_H X$. ∎

Proof of Theorem 7.7: For simplicity we consider only the case $k = 2$ and assume that $\dim_u \mu_1 > \dim_u \mu_2$. Let Λ be the set constructed in the proof of Theorem 7.6. It follows from (21), (22), (23), and the construction of the set Λ that if $x \in \Lambda$, and $x \in C^s \in \mathfrak{D}_s$ for each $s \geq 1$, then

$$\liminf_{q\to\infty} -\frac{\log \mu_2(C^{2q+1})}{S_{|C^{2q+1}|}u(x)} \geq \dim_u \mu_1 - 2\delta$$

and

$$\limsup_{q\to\infty} -\frac{\log \mu_2(C^{2q})}{S_{|C^{2q}|}u(x)} \leq \dim_u \mu_2 + 2\delta,$$

with the notation in the proof of Theorem 7.6. This implies that

$$\mathfrak{F}(\{-\log \mu_2(C_n(\cdot))/S_n u\}_{n\in\mathbb{N}}) \supset \Lambda.$$

The desired result follows from (35) and the arbitrariness of δ. ∎

8.2. Proofs of the results in Section 2.

Proof of Theorem 2.1: The remarks after Theorem 2.1 imply that we only need to show that Property 5 follows from Property 1, which in turn is an immediate consequence of Theorem 7.1. ∎

Proof of Proposition 2.3: Let $g \in L$. If there exist $n \in \mathbb{N}$ and points $x, y \in \Sigma$ such that $\sigma^n x = x$, $\sigma^n y = y$, and $S_n g(x) \neq S_n g(y)$, then g is non-cohomologous to 0. Otherwise $S_n g(x) = S_n g(y)$ for any $x, y \in \Sigma$ such that $\sigma^n x = x$ and $\sigma^n y = y$. In this case the Livshitz theorem implies that g is cohomologous to 0. Given $\varepsilon > 0$ one can find a function h in L which is ε-close to g (with respect to the supremum norm) simply by changing slightly the value of g in a small cylinder that contains the orbit of only one of the points x and y, so that $S_n h(x) \neq S_n h(y)$. Therefore h is non-cohomologous to 0. We conclude that in any neighborhood of a function in L there exist functions in L which are non-cohomologous to 0. Furthermore, if $g \in C_\theta(\Sigma)$ is non-cohomologous to 0, then any sufficiently small $C_\theta(\Sigma)$-neighborhood of g is composed by functions which are non-cohomologous to 0. This completes the proof of the proposition. ∎

Proof of Proposition 2.2: It is immediate from Proposition 2.3. ∎

Proof of Proposition 2.4: Observe that $\mathfrak{H}(\mu) \subset \mathfrak{B}$ whenever μ is a Gibbs measure (if μ is a Gibbs measure for the continuous potential ϕ, one considers the function $g = P_{\sigma|\Sigma}(\phi) - \phi$ in (4)). Hence, if $\mathfrak{A} \subset \Sigma$ is the set defined by the right-hand side of (6), then $\mathfrak{A} \subset \mathfrak{B}$.

Let $x \in \mathfrak{A}$. Then the limit $\lim_{n\to\infty} S_n\phi(x)/n$ exists for every Hölder continuous function ϕ on Σ. For a given continuous function g on Σ let $\{\phi_m\}_{m\in\mathbb{N}}$ be a sequence of Hölder continuous functions on Σ such that $\|g-\phi_m\| \to 0$ as $m \to \infty$, where $\|\cdot\|$ denotes the supremum norm on Σ. This implies that

$$\begin{aligned}
0 &\leq \limsup_{n\to\infty} \frac{1}{n} S_n g(x) - \liminf_{n\to\infty} \frac{1}{n} S_n g(x) \\
&\leq \limsup_{n\to\infty} \frac{1}{n} S_n \phi_m(x) - \liminf_{n\to\infty} \frac{1}{n} S_n \phi_m(x) + 2\|g - \phi_m\| \to 0
\end{aligned}$$

as $m \to \infty$, and hence, $x \in \mathfrak{B}$. This implies that $\mathfrak{B} \subset \mathfrak{A}$, and hence $\mathfrak{A} = \mathfrak{B}$. ∎

8.3. Proofs of the results in Sections 3 and 4.

Proof of Theorem 3.2: We start with an auxiliary result.

LEMMA 8.4: *If J is a repeller of a topologically mixing C^1 expanding map f, and R is a Markov partition, then $h(f|\partial R) < h(f|J)$.*

Proof of the lemma: The partition R is a generating partition and hence the diameter of each cylinder tends (uniformly) to zero. Therefore there exist $n \in \mathbb{N}$

and $C \in \bigvee_{k=0}^{n} f^{-k}R$ such that $C \cap \partial R = \emptyset$. Since $f(\partial R) \subset \partial R$, when we look at the coding of the boundary in the symbolic dynamics we conclude that it does not contain at least the cylinder $\chi^{-1}C$. Therefore $h(f|\partial R) < h(f|J)$. ∎

For the Markov partition R it follows from Lemma 8.4 that

$$h(f|\bigcup_{n=1}^{\infty} f^{-n}\partial R) < h(f|J),$$

since the entropy of a set coincides with the entropy of its invariant hull. Since χ is a homeomorphism on the set $J \setminus \bigcup_{n=1}^{\infty} f^{-n}\partial R$, if $A \subset \Sigma$ and $h(\sigma|A) = h(\sigma|\Sigma)$, then

$$h(f|\chi(A)) = h(f|J) = h(\sigma|\Sigma).$$

The desired statements are thus immediate consequences of Theorem 2.1. ∎

We note that a more general statement is proved in [17] for $C^{1+\varepsilon}$ expanding maps: the coding map preserves the entropy of any subset (and not only the entropy of the irregular and of the full set).

Proof of Theorem 3.3: Let ϕ be a potential for μ with $P(\phi) = 0$. The identities follow from Theorem 7.5 taking respectively $u \equiv 1$ and $u = \log a$ (see the examples after Theorem 6.2), and for each statement the sequences of functions:

1. $\{S_n g/n\}_{n\in\mathbb{N}}$, where g is some Hölder continuous function non-cohomologous to 0, since $\mathfrak{B}_f \supset \chi(\mathfrak{B}(g)) = \chi(\mathfrak{F}(\{S_n g/n\}_{n\in\mathbb{N}}))$ (the first statement also follows from Theorem 2.1);
2. $\{S_n \log a/n\}_{n\in\mathbb{N}}$;
3. $\{S_n\phi/S_n \log a\}_{n\in\mathbb{N}}$;
4. $\{S_n\phi/n\}_{n\in\mathbb{N}}$;
5. $\{S_n\phi/S_n \log a\}_{n\in\mathbb{N}}$, $\{S_n\phi/n\}_{n\in\mathbb{N}}$, and $\{S_n \log a/n\}_{n\in\mathbb{N}}$. ∎

Proof of Theorem 4.1: One can easily obtain a version of Lemma 8.4 for hyperbolic sets. With similar arguments to those in the proof of Theorem 3.2 the desired statements are thus immediate consequences of Theorem 2.1. ∎

Proof of Theorem 4.2: The case of hyperbolic sets can be reduced to that of repellers in the following way. If g is a Hölder continuous function on the two-sided subshift Σ, then there is a cohomologous function g_+ such that (see [8]):

$$g_+(\cdots i_{-1}i_0i_1\cdots) = g(\cdots i'_{-1}i'_0i'_1\cdots)$$

if $i_k = i'_k$ for every $k \geq 0$. Let Σ^+ be the one-sided subshift having the same transfer matrix as Σ. Then the irregular set $\mathfrak{B}(g_+)$ with respect to Σ^+ coincides with $\mathfrak{B}(g)$.

We now address Statement 4. We decompose Λ into local stable and unstable manifolds. For μ-almost every point $x \in \Lambda$ one can define conditional measures μ_x^s and μ_x^u on the local stable and unstable manifolds of x. There is a positive constant κ such that $\kappa^{-1}\mu(A) < (\mu_x^s \times \mu_x^u)(A) < \kappa\mu(A)$ for every measurable set A in a small rectangle. The measures μ_x^s and μ_x^u are Gibbs measures corresponding to some potentials ϕ_x^s and ϕ_x^u. See [5] and [13] for details. Since $\mu \notin \mathfrak{M}_D$, the measures μ_x^s and μ_x^u cannot both be equivalent to the measures of maximal dimension on the stable and unstable manifolds of x, respectively. Without loss of generality we assume that μ_x^u is not equivalent to the measure of maximal dimension on the unstable manifold.

Let $\mathfrak{D}_x^u(\mu)$ be the set of points in $W^u(x)$ such that the pointwise dimension of μ_x^u does not exist; $\mathfrak{D}_x^u(\mu)$ is the image under χ of the set of points $y \in \Sigma$ such that $(S_n\phi_x^u/S_n \log a^u)(y)$ does not converge. One can define $\mathfrak{D}_x^s(\mu)$ in a similar way. Proceeding as in the proof of Theorem 3.3 one can use Theorem 7.5 to obtain

$$h(f|\mathfrak{D}_x^u(\mu)) = h(f|\mathfrak{F}(\{S_n\phi_x^u/S_n \log a^u\}_{n\in\mathbb{N}})) = h(f|W^u(x) \cap \Lambda)$$

and

$$\begin{aligned}\dim_H \mathfrak{D}_x^u(\mu) &= \dim_{\log a^u} \mathfrak{F}(\{S_n\phi_x^u/S_n \log a^u\}_{n\in\mathbb{N}})\\ &= \dim_H(W^u(x) \cap \Lambda).\end{aligned}$$

Let $m_{D,x}^s$ be the measure of maximal dimension on $W^s(x)$, i.e., the Gibbs measure of $d^s \log a^s$. One can easily show that $\mathfrak{D}_y^u(\mu) \subset \mathfrak{D}(\mu)$ for $y \in W^s(x)$ in a set G_x^D of full $m_{D,x}^s$-measure. Thus, the set $\bigcup_{y\in G_x^D} \mathfrak{D}_y^u(\mu)$ is contained in $\mathfrak{D}(\mu)$ and has full stable and unstable dimensions. We obtain the second identity in Statement 4.

Let $m_{E,x}^s$ be the measure of maximal entropy on $W^s(x)$, and G_x^E a set of full $m_{E,x}^s$-measure such that $\mathfrak{D}_y^u(\mu) \subset \mathfrak{D}(\mu)$ for every $y \in G_x^E$. The set $\bigcup_{y\in G_x^E} \mathfrak{D}_y^u(\mu)$ has full topological entropy with respect to f. Hence

$$h(f|\mathfrak{D}(\mu)) \geq h(f|\bigcup_{y\in G_x^E} \mathfrak{D}_y^u(\mu)) = h(f|\Lambda).$$

This completes the proof of Statement 4.

The proofs of the remaining statements are similar to the proofs of those in Theorem 3.3. ∎

Proof of Theorem 4.5: We can consider a straightforward modification of the definition of u-dimension similar to the modification used to obtain the two-sided entropy h^* from the topological entropy. Namely, for each set $Z \subset X$ and each real number α, we define

$$M^*(Z,\alpha,u,\mathfrak{U}) = \lim_{n\to\infty} \inf_{\Gamma} \sum_{(\mathbf{U},\mathbf{V})\in\Gamma} \exp[-\alpha u(\mathbf{U}) - \alpha u(\mathbf{V})],$$

where the infimum is taken over all finite or countable collections

$$\Gamma \subset \coprod_{k+\ell>n} \mathfrak{W}_k(\mathfrak{U}) \times \mathfrak{W}_\ell(\mathfrak{U})$$

such that

$$\bigcup_{(\mathbf{U},\mathbf{V})\in\Gamma} X(\mathbf{U}) \cap f^{m(\mathbf{V})} X(\mathbf{V}) \supset Z.$$

Hence, we can set $\dim^*_{u,\mathfrak{U}} Z = \inf\{\alpha\colon M^*(Z,\alpha,u,\mathfrak{U}) = 0\}$ and the following limit exists:

$$\dim^*_u Z \overset{\text{def}}{=} \lim_{\operatorname{diam}\mathfrak{U}\to 0} \dim^*_{u,\mathfrak{U}} Z.$$

Using this "modified" u-dimension and its corresponding lower and upper pointwise dimensions, one obtains the desired statement by reproducing with minor changes the proof of Theorem 7.6; see also the discussion after the proof of Theorem 7.6. ∎

8.4. Proofs of the results in Section 6.

Proof of Theorem 6.1: This is a slight modification of the proof of Proposition 2.8 in [8]. Let $\mathfrak{V}$ be a finite open cover of X with diameter smaller than the Lebesgue number of the cover $\mathfrak{U}$. Each element $V \in \mathfrak{V}$ is contained in some element $U(V) \in \mathfrak{U}$. We write $\mathbf{U}(\mathbf{V}) = U(V_0)\cdots U(V_n)$ for each $\mathbf{V} \in \mathfrak{W}_n(\mathfrak{V})$ and observe that if $\Gamma \subset \bigcup_{k\in\mathbb{N}} \mathfrak{W}_k(\mathfrak{V})$ is a cover of Z, then $\{\mathbf{U}(\mathbf{V})\colon \mathbf{V}\in\Gamma\} \subset \bigcup_{k\in\mathbb{N}} \mathfrak{W}_n(\mathfrak{U})$ is also a cover of Z.

Set $\gamma(\mathfrak{U}) = \sup\{|u(x)-u(y)|\colon x,\, y \in U \text{ for some } U \in \mathfrak{U}\}$ and $\underline{u} = \min_{x\in X} u(x)$. We obtain

$$u(\mathbf{U}(\mathbf{V})) \le u(\mathbf{V}) + \gamma(\mathfrak{U})m(\mathbf{V}) \le (1+\gamma(\mathfrak{U})/\underline{u})u(\mathbf{V}),$$

and

$$M(Z,\alpha,u,\mathfrak{U}) \ge M(Z,(1+\gamma(\mathfrak{U})/\underline{u})\alpha,u,\mathfrak{V})$$

for each $\alpha \ge 0$. Therefore, $\dim_{u,\mathfrak{V}} Z \le (1+\gamma(\mathfrak{U})/\underline{u}) \dim_{u,\mathfrak{U}} Z$ and

$$\limsup_{\operatorname{diam}\mathfrak{V}\to 0} \dim_{u,\mathfrak{V}} Z \le (1+\gamma(\mathfrak{U})/\underline{u}) \dim_{u,\mathfrak{U}} Z.$$

By the uniform continuity of u on X, we conclude that

$$\limsup_{\operatorname{diam}\mathfrak{V}\to 0} \dim_{u,\mathfrak{V}} Z \le \liminf_{\operatorname{diam}\mathfrak{U}\to 0} \dim_{u,\mathfrak{U}} Z,$$

and $\dim_u Z$ is well defined. The proofs of the other statements are similar. ∎

Proof of Proposition 6.5: Set $\mathfrak{G} = \bigcup_\mu \mathfrak{G}(\mu)$. By the variational principle for the topological pressure [14], and Theorem 6.3, for every $\alpha \in \mathbb{R}$,

$$\begin{aligned} P_{\mathfrak{G}}(-\alpha u) &= \sup_{\text{ergodic } \mu \in \mathfrak{M}} \left(h_\mu(\sigma) - \alpha \int_X u\, d\mu \right) \\ &= \sup_{\text{ergodic } \mu \in \mathfrak{M}} \left[(\dim_u \mu - \alpha) \int_X u\, d\mu \right]. \end{aligned}$$

By Proposition 6.4, if $\alpha = \dim_u \mathfrak{G}$, then $P_{\mathfrak{G}}(-\alpha u) = 0$. Since u is positive, this happens if and only if α coincides with the right-hand side of (13), because $\int_X u\, d\mu > \min_{x\in X} u(x) > 0$ for every ergodic $\mu \in \mathfrak{M}$. ∎

Proof of Theorem 6.7: The first two statements follow easily from results in [16].

In [16], Schmeling proved that if $q \le 0$ then

$$\dim_u\{x \in \Sigma\colon \overline{d}_{\mu,u}(x) = \alpha_u(q)\} \le \mathfrak{D}_u(\alpha_u(q)),$$

and if $q \ge 0$ then

$$\dim_u\{x \in \Sigma\colon \underline{d}_{\mu,u}(x) = \alpha_u(q)\} \le \mathfrak{D}_u(\alpha_u(q)).$$

This implies that $\dim_u K_{a_1,a_2} \le \min\{\mathfrak{D}_u(a_1), \mathfrak{D}_u(a_2)\}$. One can obtain the reverse inequality by considering the distinguishing collection of measures ν_{q_1} and ν_{q_2}, where $\alpha_u(q_i) = a_i$ for $i = 1, 2$, and applying Theorem 7.2. This completes the proof of the third statement. ∎

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